Mathematics 108 Mechanics Solutions

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Solutions and Comments for the Problems

Problem Set 1

1. Let t_1, t_2 be the times of the walk and the cycle ride respectively. Since $d = vt$ we get two expressions for the distance, those being $d = 4t_1 = 12t_2$, whence $t_1 = 3t_2$. We also have $t_1 - t_2 = \frac{1}{6}$ hours. Hence $3t_2 - t_2 = \frac{1}{6} \Rightarrow t_2 = \frac{1}{12}$ hours. Therefore the distance is $12 \times \frac{1}{12}$ =1 mile.

Comment The difficulties with this problem is that the student needs to introduce his or her own symbols to stand for the quantities that are the subject of the equations that are implicit in the information offered. The other source of mistakes is in confusing the units involved as the speeds are in miles per *hour* and the times are in *minutes*. You need to work with one time unit or the other. If you don't, false factors of 60 will emerge in your answer. This kind of mistake can be detected by asking yourself whether your answer looks sensible given the facts of the problem. A moment's thought will reveal that you must have made a slip somewhere. The problem is simple enough to argue other ways: the bike is 3 times as fast as walking so when the bike arrives, the walker has $\frac{2}{3}$ of the journey to go and that takes 10 minutes. Hence the walking time is $10 \times \frac{3}{2} = 15$ minutes: in $\frac{1}{4}$ hour, the walker travels $\frac{1}{4} \times 4 = 1$ mile.

2. Speed from X to Y is 35, from Y to X is 25 km/hr. Let t_1, t_2 denote the respective times of outward and homeward journeys. Then $35t_1 = 25t_2 \Rightarrow t_2 = \frac{7}{5}t_1$. Also $t_1 + t_2 = 2$, from which we see by substitution that $\frac{12}{5}t_1 = 2 \Rightarrow t_1 = \frac{5}{6}$ hours. The distance from X to Y is thus:

$$\frac{5}{6} \times 35 = \frac{175}{6} = 29\frac{1}{6}\text{km}.$$

3. As you set out you will first meet the five trains travelling towards you that have already left London and you will also meet the next five that leave while you are travelling (the final one will just be pulling out as you arrive), so the answer is $5 + 5 = 10$. If the journey took two hours, this would double the number of trains leaving per unit of travel time and you would meet $2 \times (5+5) = 20$ trains going the other way.

Comment These calculations explain why you always meet surprisingly many trains going the other way, which makes you wonder why it often turns out that you have to wait so long for the next train when you are standing on a cold station platform. It applies to all kinds of traffic—there always seem to be lots of cars and pedestrians going past you in the opposite direction. Another similar source of confusion is that although a car may roll off the assembly line every two minutes, it does not follow that it takes two minutes to assemble a car. It may take many hours, it is just that hundreds are simultaneously being assembled at once, with them all being at different stages of production at any given moment.

4. The relative speeds of the trains is $25 + 25 = 50$mph so the collision will occur in 1 hour, in which time the fly will have flown 50 miles.

5. Differentiating $x^2 + y^2 = 25$ with respect to *time* t using the Chain Rule gives:

$$2x\dot{x} + 2y\dot{y} = 0 \Rightarrow \dot{y} = \big(-\frac{x}{y}\big)\dot{x}.$$

When $x = 4$, $y = 3, \dot{x} = 2$ (given), so that $y|_{x=4} = \big(-\frac{4}{3}\big)(2) = -\frac{8}{3}$ m/sec.

6. Solve

$$2m_0 = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow \sqrt{1 - (v^2/c^2)} = \frac{1}{2} \Rightarrow \frac{v^2}{c^2} = \frac{3}{4} \Rightarrow v = \frac{\sqrt{3}}{2}c.$$

Comment And so the mass doubles as its velocity approaches 87% of the speed of light.

7. We have

$$x(t) = A\sin nt + B\cos nt \Rightarrow \dot{x}(t) = An\cos nt - Bn\sin nt$$

$$\Rightarrow \ddot{x}(t) = -An^2\sin nt - Bn^2\cos nt = -n^2x(t).$$

8. We have $A\sin nt + B\cos nt = R\cos(x - \alpha)$ where $R = \sqrt{A^2 + B^2}$ and we may put $\alpha = \tan^{-1}\big(\frac{B}{A}\big)$. Hence the interval of motion has endpoints $\pm\sqrt{A^2 + B^2}$. The maximum first occurs when $x = \alpha = \arctan(\frac{B}{A})$. The minimum first occurs when $x = \alpha + \pi = \arctan\big(\frac{B}{A}\big) + \pi$.

9. The nearest thing to a 'straight line' is the path ABH, where B is on the rim of the glass and is such that the line ABH has no kink in it. This has length L given by

$$L^2 = 4^2 + 3^2 = 25,$$

so that $L = 5$ inches.

10. After the passing of midday, up to and including midnight, the direction of the hands coincide on 11 occasions (yes, 11, not 12) all equally spaced in time, and so $\frac{12}{11} = 1\frac{1}{11}$ hours elapse between two successive coincidence of direction.

Comment Alternatively, the minute hand runs at 1 cycle/hour while the hour hand runs at $\frac{1}{12}$ cycles/hour. The first time t in hours when the minute hand 'laps' the hour hand is when $t = 1 + \frac{1}{12}t$, which gives $\frac{11}{12}t = 1$, which is to say that $t = \frac{12}{11}$ hours. A third 'hare and tortoise' approach will draw you into a problem involving summing an infinite geometric series. See my book *Mathematics for the Curious* (Oxford University Press, 1998).

Problem Set 2

1. From $v = u + at$ we get $v^2 = (u + at)^2 = u^2 + a^2t^2 + 2uat$. From $s = \frac{1}{2}at^2 + ut$ we get $2as = a^2t^2 + 2uat$; substituting this into the previous equation then results in $v^2 = u^2 + 2as$.

2. Use $s = \frac{1}{2}at^2 + ut$ and put $u = 0$, $t = 20$ and $s = 500$ to obtain

$$500 = \frac{a}{2}(20)^2 \Rightarrow a = \frac{2 \times 500}{400} = 2 \cdot 5\text{m/sec}^2.$$

3. We use $v^2 = u^2 + 2as$, where here $a = 1$, $s = 50$ and $u = 0$ giving $v^2 = 2 \cdot 1 \cdot 50 = 100$, so that $v = 10\text{m/sec}$.

Comment This question was put to a team on University Challenge who got it wrong - it's not easy to see through a calculation involving a square root when you are put on the spot!

4. The average speed in m/sec of the cheetah is

$$\frac{u+v}{2} = \frac{6 \cdot 20 + 23 \cdot 1}{2} = 14 \cdot 65$$

and so the ground covered is $3 \cdot 3 \times 14 \cdot 65 = 48 \cdot 35\text{m}$.

5. We have $u = 7 \cdot 7$, $a = -9.81$ and at the top of the toss, $v = 0$. Hence the maximum height s reached by the ball is given by

$$0^2 = (7 \cdot 7)^2 - 2(9 \cdot 81)s \Rightarrow s = \frac{(7.7)^2}{2(9 \cdot 81)} = 3 \cdot 02\text{m}.$$

6. Put $s = 0$ in $s = \frac{1}{2}at^2 + ut$ and dividing through by t ($t = 0$ is the trivial initial solution) we obtain

$$u - \frac{1}{2}gt = 0 \Rightarrow t = \frac{2u}{g} = \frac{2(7 \cdot 7)}{9.81} = 1 \cdot 57\,\text{sec}.$$

7. Putting $s = 0$ and $t = 6$ in $s = \frac{1}{2}at^2 + ut$ gives $u - 3g = 0 \Rightarrow u = -3g$. Maximum height occurs when $v = 0$ and so from $v = u + at$ we obtain $t = \frac{-u}{-g} = \frac{3g}{g} = 3$; at which point we have

$$s = \frac{1}{2}g(3^2) - (3g)(3) = \frac{9}{2}g - 9g = -\frac{9}{2}g = (-4.5)(-9.81) = 44.1\text{m}.$$

8. From the first formula we get

$$t = \frac{x}{V\cos\alpha}$$

so that

$$y = (V\sin\alpha) \cdot \frac{x}{(V\cos\alpha)} - \frac{1}{2}g\left(\frac{x^2}{V^2\cos^2\alpha}\right)$$
$$= x\tan\alpha - \frac{gx^2\sec^2\alpha}{2V^2}.$$

9. Applying the equation of Question 8 we get

$$y = x\tan 45° - \frac{gx^2 z\sec 45°}{2(30)^2} = x - \frac{10\sqrt{2}^2 x^2}{2 \times 900} = x - \frac{x^2}{90}$$

$$\Rightarrow y' = 1 - \frac{x}{45}.$$

The horizontal component is $30\sin 45^\circ = 15\sqrt{2}ms^{-1}$. Given that $x = 30$, then $y' = 1 - \frac{30}{45} = \frac{1}{3} = \tan\theta$. The direction of travel is thus $\tan^{-1}(1/3)$ above the horizontal. Replacing $30m$ by $50m$ we get $y' = 1 - \frac{50}{45} = -\frac{1}{9}$ so that the angle is $\tan^{-1}(\frac{1}{9})$ below the horizontal.

10. The equation of Question 8 gives

$$y = x\tan\alpha - \frac{3x^2}{8h}(1 + \tan^2\alpha).$$

At $(h, h/8)$ this gives:

$$3\tan^2\alpha - 8\tan\alpha + 4 = (3\tan\alpha - 2)(\tan\alpha - 2) = 0,$$

which has solutions $\tan^{-1}(2/3)$ and $\tan^{-1} 2$.

Problem Set 3

1. Using $F = ma$ we get $F = 2 \times 10^3 \times 2\cdot 5 = 5 \times 10^3$N.

2. We have $\ddot{x} = a$ say so that $\dot{x} = at$ as $\dot{x} = 0$ when $t = 0$. But $\dot{x} = 36$km/hr $= \frac{36000}{60\times 60} = 10$m/sec when $t = 10$ so that $10 = 10a$, which is $a = 1$m/sec^2. Hence $F = ma = 2 \times 10^3 \times 1 = 2 \times 10^3$N.

3. Taking the downward direction as positive and let the force on the scales be denoted by F. The net force on the person in the positive (upward) direction is then, by Newton's 3rd law,

$$F - mg = mf \Rightarrow F = m(g + f).$$

4. Let B denote the balloon's buoyancy. The net force on the balloon is then $B - m_1 g = m_1 f$ and hence $B = m_1(g + f)$.

5. The buoyancy of the balloon in unchanged but its mass is now $m_1 - m_2$ and so the net force F on the balloon is $F = B - (m_1 - m_2)g$ upwards, and so, using the result of Question 4:

$$F = B - (m_1 - m_2)g = m_1(g + f) - (m_1 - m_2)g = m_1 f + m_2 g.$$

Hence the new acceleration equals $\frac{\text{force}}{\text{mass}} = \frac{m_1 f + m_2 g}{m_1 - m_2}$.

6. Let the y-axis be directed vertically upwards and let the unknown height by h and put $t = 0$ when the stone is dropped. We have $\ddot{y} - g$ so that $\dot{y} = c - gt$ say. Since $\dot{y} = 10$ when $t = 0$ we have $c = 10$. Hence $\dot{y} = 10 - gt$ and so $y = k + 10t - \frac{1}{2}gt^2$. But $y = h$ when $t = 0$ and so $k = h$;

$$\therefore y = h + 10t - \frac{1}{2}gt^2.$$

Now $y = 0$ when $t = 8$ and so substituting accordingly we obtain

$$0 = h + 80 - 32g \Rightarrow h = 32g - 80 \approx 233 \cdot 9\text{m}.$$

7. Let B denote the buoyancy force acting on the balloon and the ejected ballast be m. By Newton's 2nd law we have initial and final equations:

$$Mg - B = Mf_1, \ B - (M-m)g = (M-m)f_2;$$

adding these equations gives $mg = Mf_1 + (M-m)f_2$ and so

$$m = \frac{M(f_1 + f_2)}{g + f_2}.$$

8. Thrust of the aircraft in Newtons is $2 \cdot 5 \times 10^5 + 6 \times 10^3 t$. By Newton's law

$$2 \cdot 5 \times 10^5 + 6 \times 10^3 t = 10^5 \ddot{x}$$

$$\Rightarrow \ddot{x} = 2 \cdot 5 + 0 \cdot 06t \Rightarrow \dot{x} = 2 \cdot 5t + 0 \cdot 03t^2 \text{ (as } \dot{x} = 0, \text{ when } t = 0)$$

and so since $\dot{x} = 180\text{km/hour} = \frac{180 \times 1000}{60 \times 60} = 50\text{m/sec}$ at the required time t we obtain

$$\Rightarrow 50 = 2 \cdot 5t + 0 \cdot 03t^2 \Rightarrow t = \frac{-2 \cdot 5 \pm \sqrt{6 \cdot 25 + 6}}{0 \cdot 06}$$

and it is the positive root that is relevant, giving us

$$t = \frac{-2 \cdot 5 \pm 3 \cdot 5}{0 \cdot 06} = \frac{100}{6} = \frac{50}{3} \text{ seconds.}$$

9. We put y-axis vertically downwards and set $y = 0$ and $\dot{y} = 0$ when $t = 0$ (when object was dropped). Let $y = h$ denote the level of the window sill. Equation of motion is $\ddot{y} = g \Rightarrow \dot{y} = gt \Rightarrow y = \frac{1}{2}gt^2$. Let the respective times when object reaches the top and the bottom of the window by $t_1$and $t_2$when $y = h - 2$ and $y = h$ respectively. Then

$$h - 2 = \frac{1}{2}gt_1^2 \Rightarrow t_1 = \sqrt{\frac{2(h-2)}{g}}$$

$$h = \frac{1}{2}gt_2^2 \Rightarrow t_2 = \sqrt{\frac{2h}{g}}.$$

Thus

$$\frac{1}{12} = t_1 - t_1 = \sqrt{\frac{2h}{g}} - \sqrt{\frac{2(h-2)}{g}}$$

and we are asked to find h. Square and multiply by g:

$$\frac{g}{144} = 2h - 2\sqrt{4h(h-2)} + 2(h-2)$$

$$\Rightarrow 2\sqrt{4h(h-2)} = 4h - 4 - \frac{g}{144}; \text{ square again:}$$

$$16h^2 - 32h = 16h^2 - 8h(4 + \frac{g}{144}) + (4 + \frac{g}{144})^2$$

$$\Rightarrow \frac{8g}{144}h = (4 + \frac{g}{144})^2 \Rightarrow h \approx 30 \cdot 36\text{m}.$$

Comment Alternatively, it is quicker to use an average speed argument as follows. The mean speed of the object as it passes the window is $2/(1/12) = 24$m/sec, which will be attained at the window's midpoint, M. The time t taken to fall to M satisfies $gt = 24$ so that $t = \frac{24}{g}$. The average speed of the object as it falls to M is $24/2 = 12$m/sec. Hence the distance $h - 1$ fallen to reach M satsifies

$$h - 1 = 12t = \frac{12 \times 24}{g} \Rightarrow h = 1 + \frac{288}{g};$$

and so $h = 1 + \frac{288}{9 \cdot 81} = 30 \cdot 36$m.

10. As a function of time your velocity is $40 - t$ while that of the other car is $30t$. You will gain on the other car up to the point when your speeds match, which is to say $40 - t \geq 30$, that is $t \leq 10$. Measuring from your initial position, the positions of your car and the other are respectively $40t - \frac{1}{2}(1)t^2$ and $50 + 30t$. When $t = 10$ these return the same value of 350m, so that you do just catch the car.

Comment Alternatively, solve when the positions of yourself and the other car coincide: $40t - \frac{1}{2}t^2 = 50 + 30t$ so that $t^2 - 20t + 100 = (t - 10)^2 = 0$. Hence you catch the car at $t = 10$ but do not pass, for if you did, the other car would eventually pass you again as you are decelerating, but there is no second solution to the quadratic.

Problem Set 4

1. By Newton's law and Hooke's law, $mg = \frac{\lambda x_0}{l}$ so that $\lambda = \frac{mgl}{x_0}$ Newtons.

2. Writing $\omega = \sqrt{\frac{\lambda}{ml}}$ we consider the homogeneous equation $\ddot{x} + \omega^2 x = 0$. The characteristic roots of this equation are $\pm\omega i$, which yields the general solution $x(t) = A\cos\omega t + B\sin\omega t$ or equivalently $x(t) = C\cos(\omega t + \varepsilon)$ (A, B, C, ε constants). Our equation (1) has the form $\ddot{x} + \omega^2 x = g$ and so its general solution has the form $x(t) = C\cos(\omega t + \varepsilon) + k$ for some constant k. Substituting accordingly gives $\omega^2 k = g$ so that

$$k = \frac{g}{\omega^2} = \frac{gml}{\lambda} = \frac{gmlx_0}{mgl} = x_0.$$

Hence our general equation of motion is

$$x(t) = C\cos(\omega t + \varepsilon) + x_0.$$

3. We apply the initial conditions to our solution:

$$\dot{x}(0) = 0 \Rightarrow -\omega C \sin(\omega(0) + \varepsilon) = 0 \Rightarrow \varepsilon = 0;$$

$$a = C \cos(\omega(0)) + x_0 \Rightarrow C = a - x_0;$$

and so our solution is

$$x(t) = (a - x_0) \cos(\omega t) + x_0, \text{ where } \omega = \sqrt{\tfrac{\lambda}{ml}}. \tag{1}$$

4. The period of this solution is $\frac{2\pi}{\omega}$ and the frequency is the reciprocal of the period: $\frac{\omega}{2\pi}$. The maximum extension d of the spring first occurs when $\omega t = \frac{\pi}{2}$, which is to say at $t = \frac{\pi}{2\omega}$ and that maximum is $a - x_0 + x_0 = a$. In other words the maximum extension corresponds to the intial position from which the mass is released.

5. Writing v for $\dot{x}$ we have by the Chain rule that $v\frac{dv}{dx} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dt} = \ddot{x}$. We therefore may write (1) as

$$\ddot{x} + \omega^2 x = g \Rightarrow v\frac{dv}{dx} + \omega^2 x = g$$

$$\Rightarrow \int v\frac{dv}{dx}\, dx = \int (g - \omega^2 x)\, dx$$

$$\Rightarrow \frac{1}{2}v^2 = gx - \frac{\omega^2}{2}x^2$$

$$\Rightarrow v = \pm\sqrt{2gx - \omega^2 x^2}.$$

Comment The two opposite solutions correspond to the direction of the mass as it reaches a given point x.

6.
$$\frac{d}{dx}(\frac{1}{2}\dot{x}^2) = \dot{x}\frac{d\dot{x}}{dx} = \dot{x}\frac{d\dot{x}}{dt} \cdot \frac{dt}{dx} = \ddot{x}.$$

7.
$$W = \int_0^s F\, dx = \int_0^s m\ddot{x}\, dx = \frac{m}{2}\int_0^s \frac{d}{dx}(\dot{x}^2)\, dx$$

$$= \frac{m}{2}[\dot{x}^2]_0^s = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2.$$

8. The work done by the gravity is $mg(h_1 - h_2)$ and so by Question 7 we obtain

$$mg(h_1 - h_2) = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2$$

$$\Rightarrow mgh_1 + \frac{1}{2}mv_1^2 = mgh_2 + \frac{1}{2}mv_2^2.$$

Comment The term $\frac{1}{2}mv^2$ is called the *kinetic energy* of the moving object of mass m and the term mgh is the *potential energy* of the object that is a height h above a fixed reference point. The previous equation asserts the *conservation* *of energy* in that the sum of the kinetic and potential energies of the object falling under gravity is fixed.

9. Since here we have $F = \frac{\lambda x}{l}$ where we measure x from the end of the spring when at its natural length. Hence we have

$$W = \int_0^a \frac{\lambda x}{l}\,dx = \frac{\lambda}{2l}[x^2]_0^a = \frac{\lambda}{2l}[a^2 - 0^2] = \frac{\lambda a^2}{2l}.$$

10. We assign the zero of potential energy of the mass to ground zero. The velocity, and hence the kinetic energy of the mass is 0 both at release and at the point of closest approach. Hence the energy stored in the spring at this point equals the change in potential energy of the mass, which is

$$mg\Delta h = 50g(2 \cdot 2 - 0 \cdot 2) = 100g.$$

On the other hand the energy stored in the spring at the moment of greatest compression is, from Question 9, equal to

$$\frac{\lambda a^2}{2l} = \frac{\lambda(0 \cdot 4 - 0 \cdot 2)^2}{2(0 \cdot 4)} = \frac{\lambda}{20};$$

$$\Rightarrow 100g = \frac{\lambda}{20} \Rightarrow \lambda = 2000g \approx 1 \cdot 96 \times 10^4\text{N}.$$

Problem Set 5

1. Taking the zero of potential energy to be the initial position, the drop in potential energy for a given value of x $(1 \leq x \leq 2)$ is

$$\Delta E = -(\frac{mg(x-1)}{2} + \frac{m(x-1)}{2} \cdot g\frac{x-1}{2});$$

where the first and second terms in the brackets represent the loss of potential energy of the portion of the chain that intially hung off the table and the portion that slides from the table respectively;

$$= -\frac{mg}{2}(x - 1 + \frac{(x-1)^2}{2});$$

equating the loss of potential energy with the gain in kinetic energy of the chain gives:

$$\frac{1}{2}mv^2 = \frac{1}{2}mg(\frac{2x - 2 + x^2 - 2x + 1}{2})$$

$$\Rightarrow v^2 = \frac{g}{2}(x^2 - 1) \Rightarrow \frac{dx}{dt} = \sqrt{\frac{g}{2}}\sqrt{x^2 - 1}. \qquad (2)$$

2. Alternatively, the net downwards force on the chain is entirely due to gravity acting on the unsupported portion, and so by Newton's Second Law we obtain:

$$ma = mv\frac{dv}{dx} = \frac{mx}{2}g$$

$$\Rightarrow \int v\frac{dv}{dx}\,dx = \frac{g}{2}\int^{x} y\,dy$$

$$\Rightarrow \frac{v^2}{2} = \frac{g}{2}[\frac{y^2}{2}]^x \Rightarrow v^2 = \frac{gx^2}{2} + c;$$

now when $x = 1$, $v = 0$ so that $c = -\frac{g}{2}$ and so, as before we recover equation (3).

3. The differential equation (3) is separable and yields:

$$\int \frac{dx}{\sqrt{x^2-1}} = \sqrt{\frac{g}{2}}\int dt.$$

To evaluate the integral I on the left we substitute $x = \cosh y$ $(y \geq 0)$ so that $dx = \sinh y\,dy$ and $x^2 - 1 = \cosh^2 y - 1 = \sinh^2 y$. Hence we obtain:

$$I = \int \frac{\sinh y\,dy}{\sinh y} = y = \cosh^{-1} x;$$

hence we have

$$\cosh^{-1} x = \sqrt{\frac{g}{2}}t + c;$$

and when $t = 0$, $x = 1$ and $\cosh^{-1} 1 = 0$; thus $c = 0$ and we conclude that

$$x = \cosh(\sqrt{\frac{g}{2}}t).$$

The range of values to which this applies has lower limit of $t = 0$ and upper limit determined by $2 = \cosh(\sqrt{\frac{g}{2}}t)$, which gives $t = \sqrt{\frac{2}{g}}\cosh^{-1}(2)$.

4. For the chain falling freely from rest under gravity we would see $v_1^2 = 2g(x-1)$ which we compare with $v_2^2 = \frac{g}{2}(x^2-1)$. Solving $v_2 \leq v_1$ we obtain

$$v_2^2 \leq v_1^2 \Leftrightarrow \frac{1}{2}(x^2-1) \leq 2(x-1) \Leftrightarrow (x-1)(x+1) \leq 4(x-1)$$

$$\Leftrightarrow x \leq 3.$$

Since the solution of our orginal problems applies only to the range $1 \leq x \leq 2$, it follows that $v_2 \leq v_1$ throughout.

Comment The displacement of the falling chain increases only quadratically with time while the solution to Question 4 sees the movement of the sliding chain inrease exponentially. However, we see from Question 5 that the velocity, and hence the displacement of the sliding chain is less than that of the chain in free fall.

5. From Newton's law we obtain:

$$m_1 a = T - m_1 g, \ -m_2 a = T - m_2 g;$$

subtracting the second equation from the first then gives

$$(m_1 + m_2)a = g(m_2 - m_1)$$

$$\Rightarrow a = \frac{m_2 - m_1}{m_1 + m_2} g.$$

Substituting accordingly in the first equation gives:

$$T = m_1(a + g) = m_1 g(\frac{m_2 - m_1}{m_1 + m_2} + 1)$$

$$= m_1 g(\frac{m_2 - m_1 + m_1 + m_2}{m_1 + m_2}) = \frac{2m_1 m_2}{m_1 + m_2} g.$$

6. We measure the PE of the bead from the lowest point ($\theta = \pi$) so then PE $= mga(1 + \cos\theta)$. Now by Conservation of energy we have PE + KE is a constant c. When $\theta = 0$, $v = 0$ so that $c = 2mga$. Hence we obtain:

$$\frac{1}{2} mv^2 + mga(1 + \cos\theta) = 2mga$$

$$\Rightarrow v^2 = 2ga(1 - \cos\theta)$$

$$\therefore v = \sqrt{2ga(1 - \cos\theta)}.$$

Comment Note that component of the gravitational force normal to the circle does no work as the bead moves.

7. When the angle that the radius vector makes with the vertical is $\theta$the loss of potential energy from the intial position is $E = mg(a - a\cos\theta)$ and so while the puck is in contact with the sphere its velocity v satisfies

$$\frac{1}{2} mv^2 = mga(1 - \cos\theta) \Rightarrow v^2 = 2ga(1 - \cos\theta). \tag{3}$$

The puck will leave the sphere at the point where the apparent centrifugal force on the puck due to its rotation on the surface of the sphere matches the component of gravitational force normal to the surface, which is to say

$$\frac{mv^2}{a} = mg\cos\theta$$

and substituting from (4) we find

$$2mga(1 - \cos\theta) = mga\cos\theta \Rightarrow 2 - 2\cos\theta = \cos\theta$$

$$\Rightarrow \cos\theta = \frac{2}{3}$$

$$\therefore \theta = \cos^{-1}\frac{2}{3} \approx 48 \cdot 2°.$$

Comment The result is always the same, independently of the radius of the sphere and the mass of the puck (assuming friction is negligible).

8. The kinetic energy at the bottom of the swing is

$$E = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\omega^2;$$

E is transformed into potential energy at the top of the swing so that

$$E = mg\Delta h = mg(l - l\cos\theta)$$

$$\Rightarrow \frac{1}{2}ml^2\omega^2 = mgl(1 - \cos\theta)$$

$$\Rightarrow 1 - \cos\theta = \frac{l^2\omega^2}{2}$$

$$\therefore \cos\theta = 1 - \frac{l^2\omega^2}{2}.$$

9. The hanging bob will swing backwards and reach a stable position, with angle θ say, when the forward acceleration of the bob due to the tension T in the rod matches the acceleration a of the car. Resolving the horizontal and vertical components of T at equilibrium therefore yields the pair of equations:

$$ma = T\sin\theta,\ mg = T\cos\theta$$

$$\Rightarrow \tan\theta = \frac{a}{g}$$

$$\therefore \theta = \arctan^{-1}(\frac{a}{g}).$$

10. At its lowest point the bob has fallen a total distance of $l + (x - l)$. The loss of potential energy in the fall is equal to mgx. The energy stored in the string (see Question 9 Set 4) at this point is $\frac{\lambda}{2l}(x - l)^2$. Since the kinetic energy is zero at this point we may equate:

$$mgx = \frac{\lambda}{2l}(x^2 - 2lx + l^2)$$

$$\Rightarrow x^2 - (\frac{2mgl}{\lambda} + 2l)x + l^2 = 0$$

$$\Rightarrow x^2 - (1 + \frac{mg}{\lambda})2lx + l^2 = 0.$$

Comment The greater root of this quadratic represents the maximum extension of the string. The second solution is spurious but would apply if the string were a spring. In that case the expression for the stored energy would also hold under compression ($x < l$) and the mass was projected downwards from the spring's natural length with velocity corresponding to falling from the initial position.

However since the string will not absorb energy under compression, the mass will, neglecting frictional energy loss, return to its original position at the top of the string.

Problem Set 6

1. Can do this directly but note that $(\hat{\mathbf{r}}, \hat{\theta})$ is obtained from $(\mathbf{i}, \mathbf{j})$ by rotation through the angle θ so it follows that to invert the process we need only rotate through the angle $-\theta$. This gives

$$\mathbf{i} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\theta} \text{ and } \mathbf{j} = \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\theta}.$$

2. Using the Chain rule we obtain:

$$\dot{\hat{\mathbf{r}}} = \dot{\theta}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = \dot{\theta}\hat{\theta} \quad \dot{\hat{\theta}} = \dot{\theta}(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j}) = -\dot{\theta}\hat{\mathbf{r}}.$$

3. Using Question 2 we obtain:

$$\mathbf{v} = (\dot{r\hat{\mathbf{r}}}) = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}.$$

4.

$$\begin{aligned} \mathbf{a} = \dot{\mathbf{v}} =& \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\hat{\mathbf{r}}} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}} \\ =& \ddot{r}\hat{\mathbf{r}} + 2\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - \dot{r}\dot{\theta}^2\hat{\mathbf{r}} \\ & (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}. \end{aligned}$$

Comment The respective components are known as the *radial* and *transverse* components of acceleration.

5. From the formula in Question 3 for $\mathbf{v}$ we get that $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$. Differentiating $r = a(2 + \cos\theta)$ with respect to time gives $\dot{r} = -a\sin\theta\dot{\theta}$ so that

$$v^2 = a^2\sin^2\theta\dot{\theta}^2 + a^2(2 + \cos\theta)^2\dot{\theta}^2 = a^2(5 + 4\cos\theta)\dot{\theta}^2;$$

finally, since $\dot{\theta} > 0$ we take the positive square root to obtain

$$\dot{\theta} = \frac{v}{a\sqrt{5 + 4\cos\theta}}.$$

6. When moving in a circle at constant angular velocity $\omega = \dot{\theta}$, the acceleration a is directed towards it centre and, from Question 4, has value $a = r\omega^2$, so the centripetal force acting on the mass is $mr\omega^2$. Since $v = r\omega$ this can also be written as

$$a = r\omega^2 = \frac{r^2\omega^2}{r} = \frac{v^2}{r}.$$

7. From the general equation of Question 4 we get

$$\mathbf{a} = -l\dot{\theta}^2\hat{\mathbf{r}} + l\ddot{\theta}\hat{\theta} \text{ and the force on } P \text{ is given by:}$$

$$\mathbf{F} = -T\hat{\mathbf{r}} + mg\cos\theta\hat{\mathbf{r}} - mg\sin\theta\hat{\theta}.$$

8. Equating the transverse force components then gives:

$$ml\ddot{\theta} = -mg\sin\theta$$

$$\Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0.$$

9. Replacing $\sin\theta$ by θ in this equation gives $\ddot{\theta} + \frac{g}{l}\theta = 0$, which has general solution $\theta(t) = A\cos(\omega t + \alpha)$ where $\omega = \sqrt{\frac{g}{l}}$. The period of this solution is then $\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$.

10. Applying Question 4 and noting that $\dot{r} = \ddot{r} = \dot{\theta} = 0$ we see that the acceleration of the mass P is given by $-r\omega^2\hat{\mathbf{r}}$. Since $r = l\sin\theta$ we have that the acceleration of the bob is $l\sin\theta\omega^2$ directed from the bob towards the centre of the circle. Now the net vertical force on the bob is 0 so that $T\cos\theta = mg$, where T is the tension in the suspending string. The horizontal component of the force on the bob is $T\sin\theta = mr\omega^2$ and from these two equations we obtain:

$$\sin\theta = \frac{mr\omega^2}{T} = \frac{ml\sin\theta\cos\theta\omega^2}{mg}$$

$$\Rightarrow \cos\theta = \frac{g}{\omega^2 l}.$$

Problem Set 7

1. The number of hours ahead (for the Sun rises in the East) is

$$\frac{45}{360} \times 24 = 3$$

so it is 3 o'clock in the afternoon in Siberia.

2. The ratio of diameters of Sun to Moon is $93 : \frac{1}{4} = 372 : 1$, so the diameter of the Sun is about372 times that of the Moon.

3. Since the volume of a sphere is proportional to the cube of its radius (and of its diameter) the number of moons that could fit inside the Sun is about $372^3 \approx 51,480,000$ (over 51 million).

4. Leap year's aside, the anwer is $365 + 1 = 366$. This is because, looking at the solar system from above the North pole, the Earth rotates anti-clockwise and revolves around the Sun anti-clockwise as well. This cause the Sun to rise $\frac{1}{365}$ days (approximately 4 minutes) after a full rotation from the previous sunrise.

Comment If the Earth was rotating in the opposite sense to its revolution around the sun, we would see 367 days per year. This is why the day of the stars is four minutes shorter than the Earth's day, causing the night sky positions of the fixed stars to nonetheless change from one night to the next thgourghout the year. The same phenomenon affects the appearance of the moon, which revolves around the Earth (and rotates on its axis) every $27 \cdot 3$ days yet the *lunation* period, the time between one full moon and the next, is approximately $27 \cdot 3 + \frac{27 \cdot 3}{365} \times 27 \cdot 3 \approx 29 \cdot 3$ days. For more see my *Mathematics for the Imagination*, OUP).

5.

$$\frac{T_1^2}{R_1^3} = \frac{T_2^2}{R_2^3} \Rightarrow T_2^2 = \frac{R_2^3 T_1^2}{R_1^3};$$

whence upon putting $R_2 = 2R_1$ we obtain

$$T_2^2 = 8T_1^2 \Rightarrow T_2 = 2\sqrt{2}T_1 \approx 2.828 \text{ Earth years.}$$

6. Again by re-arranging the Kepler formula we get $R_2^3 = \frac{R_1^3 T_2^2}{T_1^2}$. Replacing T_2 by $29.5T_1$ then gives

$$R_2^3 = \big(\frac{29 \cdot 5T_1}{T_1}\big)^2 R_1^3 \Rightarrow R_2 = (29 \cdot 5)^{\frac{3}{2}} R_1$$

so that the distance of Saturn from the Sun is about $9 \cdot 6$ times that of the Earth.

Comment Which is then $9 \cdot 6 \times 9.3 \times 10^6 = 8.9 \times 10^7$ miles (890 million miles). Newton used his Universal Law of Gravity and calculus (which he invented) to explain Kepler's Laws.

7. Orbital speed of the Earth is

$$v = \frac{2\pi r}{T} = \frac{2\pi \times 1 \cdot 5 \times 10^{11}}{3 \cdot 16 \times 10^7} = 3 \cdot 0 \times 10^4 \text{m/sec} = 11,000 \, \text{km/hr}.$$

The acceleration of the Earth towards the Sun is (see Question 6, Set 6)

$$\frac{v^2}{r} \approx 0 \cdot 006 \text{ m/sec}^2.$$

Comment: which is negligible compared to the $g = 9 \cdot 81 \text{m/sec}^2$ acceleration due to the Earth's gravity at the surface.

8. The point P rotates around the axis daily in a circle of radius $r \cos\phi$ so the transverse velocity of P is

$$v = \frac{2\pi r \cos\phi}{T}$$

$$\Rightarrow \frac{v^2}{r} = \frac{4\pi^2 a}{T^2} \cdot \cos\phi = 0 \cdot 034 \cos\phi \, \text{m/sec}^2.$$

This acceleration is maximized at the equator where $\phi = 0$ and $\cos\phi = 1$, which then gives $0 \cdot 034 \text{ m/sec}^2$.

Comment Which is small compared to $g = -9 \cdot 81 \text{ m/sec}^2$, but of some significance, which is why satellite rockets are often fired from bases close to the equator.

9. Equating the centripetal force keeping the satellite in orbit with the force of gravity gives the equation:

$$mr\omega^2 = \frac{\gamma M m}{r^2} \Rightarrow r = \sqrt[3]{\frac{\gamma M}{\omega^2}};$$

substituting the giuven values for G, M and the value of ω now gives $r = 4.22 \times 10^7$ m.

Comment: if we subtract the radius of the Earth, 6.4×10^6m from this value, we find that the height of a synchronous satellite is 3.58×10^7m, which is about 22,250 miles above the surface of the planet.

10. From Question 9 we have

$$r^3 = \frac{\gamma M}{\omega^2};$$

since $T = \frac{2\pi}{\omega}$ if we measure in ω in radians/sec we infer that $\frac{T^2}{4\pi^2} = \frac{1}{\omega^2}$ so the previous equation yields

$$4\pi^2 r^3 = \gamma M T^2$$

$$\Rightarrow T = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{\gamma M}}.$$

Comment: and so $\frac{r^3}{T^2}$ is a constant for all planets, the value of the constant being a function of the star's mass, M.

Problem Set 8

1. After the collision the object of mass m has velocity v say so that of the unit mass is $-v$. Since the total momentum of the system is unchanged we obtain:

$$u = mv - v = v(m-1) \text{ and so } \frac{u}{v} = m - 1.$$

The (kinetic) energy of the system is $\frac{1}{2}u^2$ and since that is also conserved in any (elastic) collision we have a second equation:

$$\frac{1}{2}u^2 = \frac{1}{2}mv^2 + \frac{1}{2}v^2 \text{ and so } u^2 = v^2(m+1)$$

$$\Rightarrow \frac{u^2}{v^2} = m + 1.$$

Comparing our two equations and squaring the first leads to

$$(m-1)^2 = m + 1 \text{ and so } m^2 - 2m + 1 = m + 1$$

$$\Rightarrow m^2 - 3m = m(m-3) = 0.$$

Since $m \neq 0$ we deduce that $m = 3$.

2. We have by Newton's Second Law that

$$I = \int_{t_2}^{t_1} F\,dt = m\int_{t_1}^{t_2} \frac{dv}{dt}\,dt = m\int_{v_1}^{v_2} dv$$

where v_1 and v_2 are the velocities of the mass at times t_1 andt_2 respectively.

$$\Rightarrow I = mv_2 - mv_1 = p_2 - p_1 = \Delta p;$$

in words, the impulse of a force F acting on a mass m acting over a time interval $\Delta t = t_2 - t_2$ is equal to the change in momentum of the mass.

3. We have $\Delta p = (7{\cdot}50 - 6{\cdot}00)m = 1{\cdot}50m$. On the other hand $I = 3\times 4 = 12$ so that

$$I = 12 = 1\cdot 50m$$

$$\Rightarrow m = \frac{12}{1\cdot 50} = 8\text{kg}.$$

4. $\Delta p = 0\cdot 1 \times 50\cdot 0 = 5\cdot 00$ kg m/sec. Equate $\Delta p = F\Delta t$ so that

$$F = \frac{\Delta p}{\Delta t} = \frac{5\cdot 00}{5\cdot 00\times 10^{-3}} = 10^3$$

so that the force averages $1000N$ during the impact.

5. The initial momentum of the object is $3\cdot 00\times 4\cdot 00 = 12\cdot 00$ kg m/sec; $\Delta p = -5{\cdot}00\times 1{\cdot}80 = -9{\cdot}00$ so the final momentum is $12{\cdot}00 - 9{\cdot}00 = 3{\cdot}00 = mv$. Hence

$$v = \frac{3\cdot 00}{3\cdot 00} = 1\,\text{m/sec}.$$

6. Take p with $0 < p < 1$ such that $\int_0^p v\,dt = \int_p^1 v\,dt = \frac{1}{2}$. Suppose that $p \leq \frac{1}{2}$ and suppose that $\frac{dv}{dt} \leq a$ for $0 \leq t \leq p$. Then for any $t \leq p$,

$$\int_0^t \frac{dv}{ds}\,ds \leq at \Rightarrow v(t) - v(0) \leq at,$$

where s is a dummy variable introduced so that the symbol t does not have two meanings. Since $v(0) = 0$ we deduce that $v(t) \leq at$. Integrating again we obtain

$$\int_0^t \frac{dx}{ds}\,ds \leq a\int_0^t s\,ds \Rightarrow x(t) - x(0) \leq a[\frac{1}{2}s^2]_{s=0}^{s=t}$$

$$\Rightarrow x(t) \leq \frac{at^2}{2}.$$

Now putting $t = p \leq \frac{1}{2}$ gives (since $\frac{1}{p} \geq 2$ and so $\frac{1}{p^2} \geq 4$)

$$\frac{1}{2} = x(p) \leq \frac{ap^2}{2} \Rightarrow a \geq \frac{1}{p^2} \geq 4.$$

We conclude that if $p \leq \frac{1}{2}$ then the particle undergoes an acceleration of at least $a \geq 4$ during the interval $[0, p]$. Otherwise $p \geq \frac{1}{2}$ so that $q = 1 - p \leq \frac{1}{2}$. Put $u = 1 - t$. Then $du = -dt$, and when $t = 0, 1$ $u = 1, 0$. Suppose that $\frac{dv}{dt} \geq -a$ ($a \geq 0$) for all $p \leq t \leq 1$. Then $-\frac{dv}{du} \geq -a$ so that $\frac{dv}{du} \leq a$ for any $0 \leq u \leq q \leq \frac{1}{2}$. Replacing t by u and p by q in the previous calculation we get as before that $a \geq 4$ and so it follows that the particle undergoes a deceleration of at least $a \geq 4$ during the interval $[q, 1]$. Overall then, the particle must undergo and acceleration of magnitude at least 4 during the 1 second interval.

Comment This maximum value is attainable. Let the particle have an acceleration of 4m/sec^2 for the first $\frac{1}{2}$ second and the opposite deceleration for the second part of the interval. Hence the maximum velocity of the particle is $\frac{4}{2} = 2$ m/sec and the mean velocity of the particle is $\frac{2}{2} = 1$ m/sec (in both the first and second half second intervals) and so it travels 1 m from the origin in the time specified.

7.

$$P = \frac{W}{t} = \frac{Fd}{t} = \frac{880 \times 12}{22} = 80 \times 6$$

$$\therefore P = 480\,\text{watts} = \text{kg m}^2/\text{sec}^3.$$

8. The total mass being lifted is $500 + 300 = 800$ kg so the net force resisting the motion is $1200 + mg = 1200 + (800 \times 9.81) = 9048$N. The power required is then

$$P = \frac{Fd}{t} = Fv = 9048 \times 0 \cdot 2 = 1.81\ kW.$$

9. First 120 km/hr $= \frac{120 \times 1000}{3600} = 33 \cdot 33$ m/sec. $P = Rv$ (where R is the resistance force, which matches that of mechanical power being produced) so that

$$R = \frac{P}{v} = \frac{4.2 \times 10^4}{33 \cdot 33} = 1260\,\text{N}.$$

For the case of the slope, the force F down the slope is

$$F = R + mg\sin\theta = 1260 + 900 \times 9 \cdot 81\frac{1}{40} = 1260 + 221 = 1481\,\text{N}.$$

Finally,

$$v = \frac{P}{F} = \frac{42000}{1481} = 28 \cdot 4\,\text{m/sec} = 102\,\text{km/hr}.$$

10. The initial velocity $\mathbf{u}$ of the particle P can be expressed as $\mathbf{u} = 3\cos 30°\mathbf{i} - 3\sin 30°\mathbf{j} = 2 \cdot 60\mathbf{i} - 1 \cdot 50\mathbf{j}$. Let $\mathbf{v}$ denote the final velocity vector of the particle. The impulse vector is

$$2\mathbf{j} = \Delta\mathbf{p} = 2\mathbf{v} - 2\mathbf{u}$$

$$\Rightarrow \mathbf{v} = \mathbf{u} + \mathbf{j} = 2 \cdot 60\mathbf{i} - 0 \cdot 50\mathbf{j}.$$

Hence $v = \sqrt{2 \cdot 60^2 + 0 \cdot 50^2} = 2 \cdot 65$. The direction θ South of East satisfies $\tan\theta = \frac{0 \cdot 50}{2 \cdot 60} = 0 \cdot 1923$ so that $\theta \approx 11 \cdot 2°$, so that the bearing of the final vector is $101 \cdot 2°$ East of North.

Problem Set 9

1. In the case where N is the line $x = k > 0$ we have from the defining equation $OP = ePN$ that

$$r = e(k - r\cos\theta)$$

$$\Rightarrow r(1 + e\cos\theta) = ke$$

$$\therefore r = \frac{ke}{1 + e\cos\theta}.$$

If $x = -k < 0$ then

$$r = e(k + r\cos\theta)$$

$$\Rightarrow r = \frac{ek}{1 - e\cos\theta}. \tag{4}$$

2. We have

$$r = \frac{2 \cdot 5}{1 - \cos\theta}$$

so $e = 1$ (a parabola) and $ek = k = 2 \cdot 5$ so that the directrix N is the line $x = -2 \cdot 5$. Next

$$r - r\cos\theta = \frac{5}{2} \Rightarrow \sqrt{x^2 + y^2} - x = \frac{5}{2}$$

$$\Rightarrow x^2 + y^2 = x^2 + 5x + \frac{25}{4}$$

$$\Rightarrow x = \frac{1}{5}y^2 - \frac{5}{4}.$$

3. For $e = 1$ we have, since $r^2 = x^2 + y^2$ and $r\cos\theta = x$ that

$$r = r\cos\theta + ke \Rightarrow x^2 + y^2 = x^2 + 2kx + k^2$$

$$\Rightarrow y^2 = 2kx + k^2.$$

If we put $x = x' - \frac{k}{2}$ the equation of the conic becomes

$$y^2 = 2k(x' - \frac{k}{2}) + k^2 = 2kx' = 4ax',$$

where $k = 2a$. This is a parabola with turning point at the origin of the $x'y$ axes system. The directrix of the parabola is $x' - a = -2a$ so that $x' = -a$. The focus of the parabola is the origin of the xy-system, which is the point $(0 + a, 0) = (a, 0)$ of the $x'y$ axes.

4. The equation $OP = ePN$ now becomes

$$r = e(k - r\cos(\theta - \theta_0))$$

$$\Rightarrow r(1 + e\cos(\theta - \theta_0)) = ek$$

$$\Rightarrow r = \frac{ek}{1 + e\cos(\theta - \theta_0)}.$$

5. Let $P(x, y)$ be a point of the locus. Note that $2a > 2c$ so that $0 < c < a$. Then we have

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a > 0$$

$$\Rightarrow (x+c)^2 + y^2 = 4a^2 + (x-c)^2 + y^2 - 4a\sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow a - \frac{c}{a}x = \sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow a^2 + \frac{c^2x^2}{a^2} - 2cx = x^2 - 2cx + c^2 + y^2$$

$$\Rightarrow x^2(1 - \frac{c^2}{a^2}) + y^2 = a^2 - c^2$$

$$\Rightarrow x^2(\frac{a^2 - c^2}{a^2(a^2 - c^2)}) + \frac{y^2}{a^2 - c^2} = 1$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

Since $a > c$ we may write $a^2 - c^2 = b^2$ giving the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

an ellipse, centred at the origin with respective semi-major and semi-minor axes of lengths a and b (as $a > b$).

6. Similarly to Question 5 the given condition is captured by

$$|\sqrt{x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2}| = 2a > 0$$

$$\Rightarrow (x-c)^2 + y^2 + (x+c)^2 + y^2 = 4a^2 + 2\sqrt{((x-c)^2 + y^2)((x+c)^2 + y^2)}$$

$$\Rightarrow 2x^2 + 2y^2 + 2c^2 - 4a^2 = 2\sqrt{((x-c)^2 + y^2)((x+c)^2 + y^2)}$$

$$\Rightarrow (x^2 + y^2 + c^2 - 2a^2)^2 = ((x-c)^2 + y^2)((x+c)^2 + y^2)$$

$$\Rightarrow (x^2 + y^2 + c^2 - 2a^2)^2 = (x^2 + y^2 + c^2 - 2cx)(x^2 + y^2 + c^2 + 2cx)$$

$$\Rightarrow 4a^4 - 4a^2(x^2 + y^2 + c^2) = -4c^2x^2$$

$$\Rightarrow x^2 + y^2 + c^2 - \frac{c^2x^2}{a^2} = a^2$$

$$\Rightarrow x^2(1 - \frac{c^2}{a^2}) + y^2 = a^2 - c^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

$$\therefore \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $b^2 = c^2 - a^2 > 0$. This is a hyperbola centred at the origin. Re-writing its equation as

$$\frac{y^2}{x^2} = \frac{b^2}{a^2} - \frac{b^2}{x^2},$$

we see that for $|x| \rightarrow$ the curve approaches $y = \pm\frac{b}{a}x$, which therefore represent the asymptotes of this hyperbola.

7. We have

$$r(1 - e\cos\theta) = ek \Rightarrow r = e(k + r\cos\theta) = e(k + x)$$

$$\Rightarrow x^2 + y^2 = e^2(k^2 + 2kx + x^2)$$

$$\Rightarrow x^2(1 - e^2) - 2e^2kx + y^2 = e^2k^2$$

$$\Rightarrow x^2 - \frac{2e^2kx}{1 - e^2} + \frac{y^2}{1 - e^2} = \frac{e^2k^2}{1 - e^2}$$

$$\Rightarrow (x - \frac{e^2k}{1 - e^2})^2 + \frac{y^2}{1 - e^2} = \frac{2e^2k^2}{1 - e^2}; \tag{5}$$

if $e < 1$ then $1 - e^2 > 0$ and (6) represents an ellipse, as the equation can be written in the form $\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = 1$, while if $e > 1$ (6) represents a hyperbola, as the equation can be written in the form $\frac{(x-c)^2}{a^2} - \frac{y^2}{b^2} = 1$.

8. We have

$$r = \frac{l}{1 + e\cos\theta}, \ l = 0 < e < 1, \ l = ke > 0.$$

Hence the *aphelion* position A (of the planet), which is the furthest point from the Sun at the focus O, is given by

$$r_A = \frac{l}{1 - e}, \text{ when } \theta = \pi, \text{ so that } \cos\theta = -1.$$

The *perihelion* position P, which is the point where the planet is closest to the Sun, is given by

$$r_P = \frac{l}{1 + e} \text{ when } \theta = 0, \text{ so that } \cos\theta = 1.$$

The *semi-latus rectum* is the vertical distance from the focus to the curve, which corresponds to $\theta = \frac{\pi}{2}$, in which we get $r = l = ek$.

9. Writing a for the major semi-axis of the ellipse we have

$$2a = r_P + r_A = \frac{l}{1 - e} + \frac{l}{1 + e} = \frac{l(1 + e + 1 - e)}{(1 - e)(1 + e)} = \frac{2l}{1 - e^2}$$

$$\therefore a = \frac{ke}{1 - e^2} = \frac{r_A}{1 + e}. \tag{6}$$

10. Following on from Question 9, we have, from (7), writing ON for the distance from the focus to the directrix N

$$ON = k = \frac{a}{e}(1 - e^2).$$

Again from (7) we find that the distance CO, where C is the centre of the ellipse satisfies

$$CO = r_A - a = a(1 + e) - a = ae. \qquad (7)$$

Finally. denoting by B the point of the ellipse vertically above the centre C we have $BO = eBN = eCN$ so from (8) we obtain

$$BO = e(ae + k) = e(ae + \frac{a}{e} - ae) = a. \qquad (8)$$

By considering the right triangle $\triangle OBC$ and using (9) and (10) we find b, the minor semi-axis of the ellipse, as

$$b^2 + (CO)^2 = (BO)^2$$

$$\Rightarrow b^2 + (ae)^2 = a^2$$

$$\therefore b = a\sqrt{1 - e^2}.$$

Problem Set 10

1. We work backwards: the condition that $r^2\dot{\theta} = h$, a constant implies that

$$\frac{d}{dt}(r^2\dot{\theta}) = 0 \Leftrightarrow (2r\dot{r})\dot{\theta} + r^2\ddot{\theta} = 0$$

$$\Leftrightarrow 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0,$$

so that the given condition is equivalent to saying that the radial component of acceleration is 0, so that the force on the object is purely radial.

Comment it follows that for a mas m orbiting the Sun, $mr^2\ddot{\theta}$ is constant. This quantity is the *angular momentum* of the mass, a topic that we shall return to in the Second Year modules. Kepler's Second Law, which states that the rate at which a planet sweeps out radial area is constant, represents a manifestation of the law of conservation of angular momentum.

2. If P moves from P_1 to P_2 in the time interval $t_1$to t_2 then the area swept out by the the ray OP is

$$A = \frac{1}{2}\int_{\theta(t_1)}^{\theta(t_2)} r^2\, d\theta = \frac{1}{2}\int_{t_1}^{t_2} r^2 \frac{d\theta}{dt}\, dt$$

$$= \frac{1}{2}\int_{t_1}^{t_2} h\,dt = \frac{h}{2}(t_2 - t_1).$$

Hence the area swept out depends only on the length of the time interval and is independent of the value of r. This is *Kepler's Second Law*: a planet sweeps out equal areas in equal times.

3. From Question 1, $r^2\dot{\theta} = h$, a constant, and so

$$h^2 = r^4\dot{\theta}^2 = h^2 \Rightarrow r\dot{\theta}^2 = \frac{h^2}{r^3}.$$

Hence, if we now equate the gravitational force to the radial force acting on our mass m and cancel the common factor of m we obtain:

$$-\frac{\gamma M}{r^2} = \ddot{r} - r\dot{\theta}^2 = \ddot{r} - \frac{h^2}{r^3}.$$

4. By the Chain rule and Question 1 we get

$$\frac{d}{dt} = \frac{d}{d\theta}\cdot\frac{d\theta}{dt} = \dot{\theta}\frac{d}{d\theta} = \frac{h}{r^2}\frac{d}{d\theta}.$$

Substituting accordingly into our differential equation now gives:

$$-\frac{\gamma M}{r^2} = \frac{d}{dt}\big(\frac{dr}{dt}\big) - \frac{h^2}{r^3} = \frac{h}{r^2}\frac{d}{d\theta}\big(\frac{h}{r^2}\frac{dr}{d\theta}\big) - \frac{h^2}{r^3}$$

$$\Rightarrow -\frac{\gamma M}{h^2} = \frac{d}{d\theta}\big(\frac{1}{r^2}\frac{dr}{d\theta}\big) - \frac{1}{r}.$$

5. Now $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$ and so substituting into the equation of Question 4 gives:

$$-\frac{\gamma M}{h^2} = \frac{d}{d\theta}\big(-\frac{du}{d\theta}\big) - u$$

$$\therefore \frac{d^2u}{d\theta^2} + u = \frac{\gamma M}{h^2}, \text{ where } u = \tfrac{1}{r}.$$

The general solution of the corresponding homogeneous equation is $u = A'\cos\theta + B'\sin\theta$ (A', B' constants) or, if we choose, as we do here, $u = A\cos(\theta - \theta_0)$ for arbitrary constants A and θ_0. For a particular integral we may take any constant, so in particular we may have $u = \frac{\gamma M}{h^2}$, giving the general solution:

$$u(\theta) = A\cos(\theta - \theta_0) + \frac{\gamma M}{h^2}$$

$$\Rightarrow \frac{1}{r} = \frac{1 + \frac{h^2 A}{\gamma M}\cos(\theta - \theta_0)}{\frac{h^2}{\gamma M}}$$

Re-writing the solution in terms of r now gives:

$$r(\theta) = \frac{\frac{h^2}{\gamma M}}{1 + \frac{h^2 A}{\gamma M}\cos(\theta - \theta_0)};$$

this is a conic section with $e = \frac{h^2 A}{\gamma M}$ and $k = \frac{1}{A}$, one focus at the origin O and corresponding directrix at a distance $k = \frac{1}{A}$ from O with θ_0 the angle between the perpendicular to the directrix and the x-axis, measure (anti-clockwise).

Comment In particular, an orbit is either elliptical, parabolic, or hyperbolic. In the case of a planet or other body trapped in the star system, the orbit is elliptical with the Sun at one focus of the orbit, as the Sun is the chosen origin of our coordinate system. This is *Kepler's First Law*.

6. The area of the ellipse forming the orbit is πab. Using the result of Question 1 we can also express this area as:

$$\pi ab = \frac{1}{2}\int_0^{2\pi} r^2\,d\theta = \frac{1}{2}\int_0^T r^2\frac{d\theta}{dt}\,dt = \frac{h}{2}\int_0^T dt = \frac{h}{2}(T-0) = \frac{hT}{2},$$

$$\Rightarrow T = \frac{2\pi ab}{h}.$$

7. The gravitational potential V of a unit mass m at a distance r from a mass M is the work done to move m from r to infinity:

$$V = \int_r^\infty -\frac{\gamma M}{x^2}\,dx = -\gamma M[-\frac{1}{x}]_{x=r}^{x=\infty}$$

$$= -\gamma M[0 - (-\frac{1}{r})] = -\frac{\gamma M}{r}.$$

8. In general we have $h = r^2\dot{\theta}$. At perihelion, the radial component of acceleration is 0 so that the velocity is in the direction of $\hat{\theta}$ and has value $v_P = r\dot{\theta}$, whence $h = r_P v_P$. Now the energy of the planet E is constant and the equation $E = V + P$ (kinetic plus potential) at perihelion takes on the form:

$$E = \frac{1}{2}mv_P^2 - \frac{\gamma Mm}{r_P}.$$

Now we substitute $v_P = \frac{h}{r_P}$ and $r_P = \frac{l}{1+e}$ (see Question 8 Set 9) to obtain:

$$E = \frac{m}{2}\cdot\frac{h^2}{r_P^2} - \frac{\gamma Mm}{r_P}.$$

For Question 5 we have $\frac{h^2}{\gamma M} = l$ so that $h^2 = \gamma Ml$. This becomes

$$E = \frac{\gamma Mm}{2}[\frac{l(1+e)^2}{l^2} - \frac{2(1+e)}{l}] = \frac{\gamma Mm}{2l}[(1+e)^2 - 2(1+e)]$$

$$\Rightarrow E = \frac{\gamma Mm}{2l}(e^2 - 1).$$

9. It follows from Question 8 that for an elliptical orbit, ($e < 1$), we have $E < 0$, for a parabolic orbit ($e = 1$), $E = 0$ and for a hyperbolic orbit ($e > 1$), $E > 0$. In particular, a planet will escape the Sun's gravitational pull if and only if $E > 0$, which is to say

$$\frac{1}{2}mv^2 \geq \frac{\gamma M m}{r}$$

$$\Leftrightarrow v \geq \sqrt{\frac{2\gamma M}{r}}.$$

Comment This bounding speed is known as the *escape velocity* of the orbiting body.

10. From Question 5 we have $\frac{h^2}{\gamma M} = l$ so that $h^2 = \gamma M l$. Also from Question 8 on Set 9 we have $b = a\sqrt{1-e^2}$ and from Question 9 on Set 9 we have $l = b\sqrt{1-e^2}$ so that $l = a(1-e^2)$. Using these equations and Question 7 then gives:

$$T = \frac{2\pi ab}{h} = \frac{2\pi a^2\sqrt{1-e^2}}{\sqrt{\gamma M}\sqrt{a}\sqrt{1-e^2}} = \frac{2\pi}{\sqrt{\gamma M}} \cdot a^{\frac{3}{2}}$$

which is *Kepler's Third Law* for an elliptical orbit.

Comment Question 10 on Set 7 was the special case where the orbit was circular.