Mathematics 103 Calculus I Solutions

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Solutions and Comments for the Problems

Problem Set 1

1.
$$y' = 2\sec(x^2)\cdot(\sec(x^2))' = 2\sec(x^2)\cdot 2x\cdot\sec(x^2)\tan(x^2)$$
$$\Rightarrow y' = 4x\sec^2(x^2)\tan(x^2).$$

2.
$$\ln y = \ln(x^2(7x-14)^{\frac{1}{3}}) - \ln((1+x^2)^4)$$
$$= 2\ln x + \frac{1}{3}\ln(7x-14) - 4\ln(1+x^2).$$
$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \frac{2}{x} + \frac{7/3}{7x-14} - \frac{8x}{1+x^2}$$
$$\Rightarrow \frac{dy}{dx} = \frac{x^2(7x-14)^{\frac{1}{3}}}{(1+x^2)^4}\Big(\frac{2}{x} + \frac{1}{3x-6} - \frac{8x}{1+x^2}\Big).$$

Comment This technique is suitable when asked to find the derivative that involves a quotient of terms that are themselves products of several terms.

3. $y' = 3x^2-10x+1,$. $y'(1) = 3-10+1 = -6$, so that $y = -6x+c$. At $x = 1$, $y = 1-5+1+6 = 3$. Using $(1,3)$ we see that $3 = -6(1)+c \Rightarrow c = 9$. Hence the required tangent is $y = -6x+9$, which in the required form is $6x+y-9 = 0$.

4. $5y^2 + \sin y = x^2 \Rightarrow 10yy\prime + (\cos y)y\prime = 2x \Rightarrow y\prime = \frac{2x}{10y+\cos y}$.

5. Differentiating implicitly with respect to x yields:
$$28y^3y' + 3x^2y + x^3y' + 1 = 0 \Rightarrow$$
$$y' = -\frac{3yx^2+1}{28y^3+x^3}.$$
Evaluating at $(4,0)$ gives the value of the slope as $-\frac{1}{4^3} = -\frac{1}{64}$.

6.
$$(cosec(2\cot 3x))\prime = (-cosec(2\cot 3x)\cot(2\cot 3x))\cdot(2\cdot 3(-cosec^2 3x))$$
$$= 6cosec(2\cot 3x)\cot(2\cot 3x)\cdot cosec^2 3x.$$

7.
$$y\prime = -\frac{1}{\sqrt{1-x^2}} - \Big(-\frac{1}{2}(1-x^2)^{-\frac{3}{2}}\Big)\Big(-2x\Big) + (-1)(\ln 2)2^{-x}$$
$$= -\frac{1}{\sqrt{1-x^2}} - \frac{x}{(1-x^2)^{\frac{3}{2}}} - \frac{\ln 2}{2^x}.$$

8.
$$y\prime = \frac{(1-\sin x)}{\cos x}\cdot\frac{-\sin x(1-\sin x)-(-\cos^2 x)}{(1-\sin x)^2}$$
$$= \frac{-\sin x+\sin^2 x+\cos^2 x}{\cos x(1-\sin x)} = \frac{1-\sin x}{\cos x(1-\sin x)} = \sec x.$$

9. Put
$$f\prime(x) = \frac{1-\ln x}{x^2} = 0 \Leftrightarrow x = e,$$
which yields a turning point with co-ordinates (e, e^{-1}). This is a maximum: $f''(x) = \frac{(-3x+2x\ln x)}{x^4}$ and so $f''(e) = -\frac{1}{e^3} < 0$, whence the turning point is a maximum by the 2nd derivative test.

10. $f(x) = \text{constant} \Leftrightarrow f\prime(x) = 0$ and in these circumstances that gives:
$$\frac{a(cx+d)-c(ax+b)}{(cx+d)^2} = 0 \Leftrightarrow ad - bc = 0.$$
Hence $f(x)$ is a constant function if and only if $ad = bc$.

Comment When we say that $f(x)$ is the constant function c we mean that $f(x) = c\,\forall x$ in the domain of $f(x)$; for emphasis this is sometimes written as $f(x) \equiv 0$ so as not to confuse this idea with the finding of the roots of a function $f(x)$ which are the values of x, if they exist, such that $f(x) = 0$.

Problem Set 2

1. $I = \int x\cos x\,dx$. Integrate by parts with $u = x$, so that $du = dx$;$dv = \cos x\,dx$, and $v = \sin x$. Hence
$$I = x\sin x - \int \sin x\,dx = x\sin x + \cos x + c.$$

2. Put $u = \cos x$, so $du = -\sin x$; $x = 0 \Rightarrow u = 1$; $x = \frac{\pi}{4} \Rightarrow u = \frac{1}{\sqrt{2}}$. Hence the integral I becomes:
$$I = -\int_1^{\frac{1}{\sqrt{2}}}\frac{du}{u} = \int_{\frac{1}{\sqrt{2}}}^1 \frac{du}{u} = \ln u|_{u=\frac{1}{\sqrt{2}}}^{u=1}$$
$$= 0 - \ln\left(\frac{1}{\sqrt{2}}\right) = \ln 2^{\frac{1}{2}} = \frac{1}{2}\ln 2.$$
Comments When calculating a definite integral using a substitution it is normally easier to express the limits in terms of the new variable as well. Alternatively, you can find the indefinite integral, substitute back to express this in terms of the original variable and apply the original limits. However, a mixed

expression involving both your original and substituted variables will rarely be of any use to you. As usual when dealing with logs, it is important not only to know your log laws but to use them in order to simplify the expressions that arise: $\log ab = \log a + \log b$; $\log \frac{a}{b} = \log a - \log b$ and $\log a^b = b \log a$.

3. $\frac{1-x}{1+x} = -1 + \frac{2}{1+x}$. Hence

$$\int \frac{1-x}{1+x}\,dx = \int (-1 + \frac{2}{1+x})\,dx = -x + 2\int \frac{dx}{1+x} = -x + 2\ln|1+x| + c$$

4. $\sin 5x \cos 11x = \frac{1}{2}\sin(5x - 11x) + \frac{1}{2}\sin(5x + 11x)$. Hence our integral becomes:

$$\frac{1}{2}\int \sin 16x\,dx - \frac{1}{2}\int \sin 6x\,dx =$$

$$= -\frac{1}{32}\cos 16x + \frac{1}{12}\cos 6x + c.$$

5.

$$I = \int \sec^6\theta\,d\theta = \int \sec^4\theta \sec^2\theta\,d\theta = \int (1+\tan^2\theta)^2 \sec^2\theta d\theta = \int (1+u^2)^2\,du$$

(where $u = \tan\theta$), so $du = \sec^2\theta d\theta$. Hence

$$I = \int (1 + 2u^2 + u^4)\,du = u + \frac{2}{3}u^3 + \frac{1}{5}u^5 + c;$$

$$\therefore I = \tan\theta + \frac{2}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta + c.$$

6. Put $u = x - 1$, $du = dx$, giving

$$\int_{u=0}^{u=1} (u+1)^2 u^{\frac{1}{2}}\,du = \int_0^1 (u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + u^{\frac{1}{2}})\,du$$

$$= [\frac{2}{7}u^{\frac{7}{2}} + \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}]_0^1 = \frac{2}{7} + \frac{4}{5} + \frac{2}{3} = \frac{184}{105}.$$

7.

$$\frac{1}{x^2+x-2} = \frac{1}{(x-1)(x+2)} \equiv \frac{A}{x-1} + \frac{B}{x+2};$$

$$\Rightarrow A(x+2) + B(x-1) = 1.$$

Put $x = -2$: $-3B = 1 \Rightarrow B = -\frac{1}{3}$; put $x = 1$: then $3A = 1 \Rightarrow A = \frac{1}{3}$. Hence the integral I becomes:

$$I = \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{dx}{x+2} = \frac{1}{3}\Big(\ln|x-1| - \ln|x+2|\Big) + c$$

$$= \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + c.$$

Comment A quick way to split a rational function with linear factors into partial fractions is the so-called *Cauchy Cover-up method.* Take the root of each linear factor in turn, cover that term up and substitute the value of that root in the remaining expression to get the value of the numerator in the corresponding fraction. In this example, cover up the $x-1$ term in the denominator and put $x=1$ to get $A=\frac{1}{1+2}=\frac{1}{3}$; similarly cover up the $x+2$ term and put $x=-2$ to get $B=\frac{1}{-2-1}=-\frac{1}{3}$.

8. Put $u=e^x$; $du=e^x dx \Rightarrow dx=\frac{du}{u}$; the integral becomes:

$$\int \frac{du}{u(1+u)} = \int (\frac{1}{u} - \frac{1}{1+u})\,du = \ln(e^x) - \ln(1+e^x) + c =$$

$$= x - \ln(1+e^x) + c.$$

Comments Note there is no need for absolute value signs around the log arguments as $e^x > 0$ for all x. Applying the Cover Up Method to this example we have that the roots of u and $1+u$ are respectively 0 and -1: covering up the u term in $\frac{1}{u(1+u)}$ and putting $u=0$ in $\frac{1}{1+u}$ gives 1, the numerator of $\frac{1}{u}$; similarly covering up the $1+u$ term and putting $u=-1$ in $\frac{1}{u}$ returns the value of -1, which we see in the second term of the decomposition: $-\frac{1}{1+u}$.

9. Put $u=\ln x$ and $dv=dx$; we then get $du=\frac{dx}{x}$, $v=x$, and so the integral becomes

$$x\ln x - \int x\frac{dx}{x} = x\ln x - x + c.$$

10. We note that the sign of $\cos 2x$ turns from positive to negative at $x=\frac{\pi}{4}$ and so we write:

$$\int_0^{\frac{\pi}{2}} |\cos 2x|\,dx = \int_0^{\frac{\pi}{4}} |\cos 2x|\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} |\cos 2x|\,dx$$

$$= \int_0^{\frac{\pi}{4}} \cos 2x\,dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 2x\,dx$$

$$= [\frac{1}{2}\sin 2x]_0^{\frac{\pi}{4}} - [\frac{1}{2}\sin 2x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\big((\sin\frac{\pi}{2} - \sin 0) - (\sin\pi - \sin\frac{\pi}{2})\big) = \frac{1}{2}\big(1-0-(0-1)\big)$$

$$= \frac{1}{2}(1+1) = \frac{1}{2}(2) = 1.$$

Comment Alternatively we might just appeal to the fact that the form of the cosine curve below the x-axis is identical to that above and so conclude that the area represented by the integral is equal to $2\int_0^{\frac{\pi}{4}} \cos 2x\,dx = 2[\frac{1}{2}\sin 2x]_0^{\frac{\pi}{4}} = 2(\frac{1}{2}-0) = 2\cdot\frac{1}{2} = 1$ unit^2.

Problem Set 3

1. Put $y = 2x + 3$ so that $x = \frac{1}{2}(y - 3)$. Hence

$$f(y) = \frac{1}{4}(y^2 - 6y + 9) + 1 \text{ and so, as a function of } x:$$

$$f(x) = \frac{1}{4}x^2 - \frac{3}{2}x + \frac{13}{4}.$$

2. Putting $y = e^x$ we have $f(y) = \frac{1}{2}(y + \frac{1}{y})$ so, as a function of the symbol x we have

$$f(x) = \frac{x^2 + 1}{2x}.$$

3. Write $f(x) = e(x) + o(x)$ where $e(x)$ is an even function and $o(x)$ is odd. Then $f(-x) = e(-x) + o(-x) = e(x) - o(x)$. Adding and then subtracting the two equations in $f(x)$ then gives

$$e(x) = \frac{f(x) + f(-x)}{2}, \ o(x) = \frac{f(x) - f(-x)}{2}.$$

In particular, taking $f(x) = e^x$ gives that $e(x) = \cosh x$, and $o(x) = \sinh x$.

Comment The argument above strictly speaking, merely shows that $e(x)$ and $o(x)$ are unique for we have found out what they are, assuming they do exist. However the manipulations can be reversed and so $e(x)$ and $o(x)$ as defined above exist and satisfy $e(x) + o(x) = f(x)$. Moreover, from their definitions it follows that $e(-x) = e(x)$ and $o(-x) = -o(x)$, as we require.

4. In this instance the equation becomes $a^2x + b(a + 1) = 2x + 1$ so that $a = \pm\sqrt{2}$ and $b = 1/(1 \pm \sqrt{2})$:

$$f(x) = \pm\sqrt{2}x + \frac{1}{1 \pm \sqrt{2}}.$$

5. $(f \circ f)(x) = a(ax + b) + b = a^2x + b(a + 1) \equiv x$. This is equivalent to

$$a^2 = 1 \,\&\, b(a + 1) = 0 \Leftrightarrow a = \pm 1 \,\&\, b(a + 1) = 0$$

$$\Leftrightarrow a = -1 \; or \; a = 1, b = 0.$$

Hence the solutions are $f(x) = x$ and $f(x) = b - x$ for arbitrary b.

Comment Note that the graphs of these functions are precisely the straight lines that are invariant under reflection in the line $y = x$.

6. From $f(f(x)) = x^2 - x + 1$ we have in particular that $f(f(0)) = 1 = f(f(1))$ and so

$$f(1) = f(f(f(1))) = f^2(1) - f(1) + 1$$
$$\Rightarrow f^2(1) - 2f(1) + 1 = (f(1) - 1)^2 = 0 \Rightarrow f(1) = 1.$$

Next

$$f(f(f(0))) = f^2(0) - f(0) + 1 = f(1) = 1.$$

Therefore $f(1) = 1$ and so $f^2(0) - f(0) = 0$, so that either $f(0) = 0$ or $f(0) = 1$. However, if $f(0) = 0$ then $f(f(0)) = f(0) = 0$, contradicting that $f(f(0)) = 1$. Hence we conclude that $f(0) = 1$.

7. In order to invert the function we make x the subject of the formula:

$$cxy + dy - ax - b = 0 \Rightarrow x(cy - a) = b - dy \Rightarrow x = \frac{-dy + b}{cy - a}.$$

Hence the inverse function is given by the rule:

$$y = \frac{-dx + b}{cx - a}.$$

Note that the condition $ad \neq bc$ is, by Question 10 of Problem Set 1, what is needed to guarantee that $\frac{ax+b}{cx+d}$ is not a constant function, in which case we see that the same applies to the inverse. In particular, $cx - a \not\equiv 0$ as $cx - a = 0\, \forall x \Leftrightarrow a = c = 0 \Rightarrow ad = bc\, (= 0)$.

8. $\cos^{-1}(\cos x) = x\, \forall\, 0 \leq x \leq \pi$, for this interval is the principal domain of the cosine function. Since $\cos x$ is even, so is the given function $y(x)$; since $\cos x$ has period 2π, it follows that $y(x + 2\pi) = y(x)$. Collectively, this is sufficient information to sketch $y(x)$ for all values of x: we get a *saw-toothed wave* as seen in the graph below.

9. We require that

$$(2 - x > 0\, \&\, \sin x + \cos x \neq 0)$$

$$\Leftrightarrow (x < 2)\, \&\, (\sqrt{2}\sin(x + \frac{\pi}{4}) \neq 0)$$

$$\Leftrightarrow (x < 2)\, \&\, x + \frac{\pi}{4} \neq n\pi,\ n \in \mathbb{Z}$$

$$\Leftrightarrow (x < 2)\, \&\, (x \notin \{(n - \frac{1}{4})\pi,\ (n \in \mathbb{Z})\}.$$

Therefore the natural domain of $f(x)$ is:

$$(-\infty, 2) \setminus \{(n - \frac{1}{4})\pi,\ n \leq 0\}.$$

10. Since $-1 \leq \sin x \leq 1$. We require $0 < \sin x$. However, for $0 < x \leq 1$ we have $-\infty < \ln x \leq 0$, so that $\ln(\ln(\sin x))$ is not defined for any $x \in \mathbb{R}$. Hence the natural domain of this functional rule is $\emptyset$.

Comment Despite having an empty domain we can formally calculate $(\ln(\ln(\sin x))'$ and get

$$\frac{1}{\ln(\sin x)} \cdot \frac{\cos x}{\sin x} = \frac{\cot x}{\ln(\sin x)},$$

which defines a function with natural domain that is not empty.

Problem Set 4

1.

$$\ln y = -2x(\ln a) \Rightarrow \frac{y\prime}{y} = -2\ln a$$

$$\therefore y\prime = -2\ln(a) \cdot a^{-2x}.$$

Comment Alternatively, by definition, $a^x = e^{(\ln a)x}$ and go from there.

2. We want

$$\lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{\sum_{k=1}^{n} \binom{n}{k} h^k x^{n-k}}{h}$$

$$\lim_{h\to} \sum_{k=1}^{n} \binom{n}{k} h^{k-1} x^{n-k} = \binom{n}{1} x^{n-1} = nx^{n-1};$$

as all other terms contain a postive power of h and so approach 0 as $h \to 0$.

3.

$$y\prime = \left(\frac{1}{1+\left(\frac{1}{1+x^2}\right)^2}\right) \cdot \left(\frac{-2x}{(1+x^2)^2}\right) = \frac{-(1+x^2)^2}{(1+x^2)^2+1} \cdot \frac{2x}{(1+x^2)^2}$$

$$= \frac{-2x}{1+(1+x^2)^2} \text{ or } -\frac{2x}{x^4+2x^2+2}.$$

4.

$$x = \sec y \Rightarrow \frac{dx}{dy} = \sec y \tan y \Rightarrow \frac{dy}{dx} = \frac{1}{\sec y \tan y}.$$

Now $\tan^2 y = \sec^2 y - 1 = x^2 - 1$, $x = \sec y$ and so

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}} \text{ if tany} > 0.$$

If $\tan y < 0$, then $\tan y = -\sqrt{x^2-1}$and

$$\frac{dy}{dx} = \frac{1}{-x\sqrt{x^2-1}}.$$

In either case we obtain:

$$\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}.$$

5. We have $f\prime(x) = -\frac{1}{\sqrt{2\pi}} x e^{-\frac{x^2}{2}}$ and

$$f\prime\prime(x) = -\frac{1}{\sqrt{2\pi}}\left(e^{-\frac{x^2}{2}} - x^2 e^{-\frac{x^2}{2}}\right)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \big(x^2 - 1\big).$$

Hence $f''(x) = 0$ iff $x^2 = 1$, that is $x = \pm 1$, in which case $y = \frac{e^{-\frac{1}{2}}}{\sqrt{2\pi}}$. Therefore the required points are $\big(\pm 1, \frac{1}{\sqrt{2\pi e}}\big)$.

Comment These are the two points of the bell-shaped curve where we see a change of shape from concave up to concave down. The unique turning point of course occurs at the origin.

6.

$$y' = -cosec\theta \cot\theta = -\frac{\cos\theta}{\sin^2\theta}.$$

Hence

$$y'\Big(\frac{\pi}{6}\Big) = -\frac{\cos\big(\frac{\pi}{6}\big)}{\sin^2\big(\frac{\pi}{6}\big)} = -\frac{\sqrt{3}/2}{(1/2)^2} = -2\sqrt{3}.$$

Now $cosec\Big(\frac{\pi}{6}\Big) = \frac{1}{\sin\big(\frac{\pi}{6}\big)} = \frac{1}{1/2} = 2$. The gradient of the normal is then $\frac{-1}{-2\sqrt{3}} = \frac{\sqrt{3}}{6}$. The equation of the normal is: $y = \big(\frac{\sqrt{3}}{6}\big)x + c$, where

$$2 = \frac{\sqrt{3}}{6} \cdot \frac{\pi}{6} + c \Rightarrow c = 2 - \frac{\sqrt{3}\pi}{36}.$$

Therefore the equation of the normal is $y = \frac{\sqrt{3}}{6}x + (2 - \frac{\sqrt{3}\pi}{36}\big)$.

7. By the Chain Rule we obtain:

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt} = \sec^2 x \cdot (12t^2 + 1)$$

$$= \sec^2(4t^3 + t)(12t^2 + 1).$$

8. $y' = 2\cos 2x - 4\sin 4x$. Then

$$\tan\theta = y'\Big(\frac{\pi}{8}\Big) = 2\cos\frac{\pi}{4} - 4\sin\frac{\pi}{2} = \frac{2}{\sqrt{2}} - 4 = \sqrt{2} - 4.$$

9. Let R and r denote the respective radii of the cone and cylinder and similarly let H and h be their respective heights. By similar triangles we have:

$$\frac{h}{R - r} = \frac{H}{R} \Rightarrow h = \frac{H(R - r)}{R}.$$

Thus writing V_S for the volume of the cylinder we have the equation:

$$V_S(r) = \pi r^2 h = \frac{\pi H}{R}\big(Rr^2 - r^3\big)$$

$$V_S'(r) = \frac{\pi H}{R}\big(2Rr - 3r^2\big) = 0 \Rightarrow r = 0 \text{ or } r = \tfrac{2R}{3}.$$

Clearly the latter is the maximizing value, whence $V_S\left(\frac{2R}{3}\right) = \frac{4}{27}\pi R^2 H$ and the volume of the cone is then given by $V_C = \frac{1}{3}\pi R^2 H$. Hence at this optimal value we have:

$$\frac{V_C}{V_S} = \frac{\left(\frac{1}{3}\pi R^2 H\right)}{\left(\frac{4}{27}\pi R^2 H\right)} = \frac{9}{4},$$

so that the ratio $V_S : V_C = 4 : 9$.

10. By the Fundamental Theorem of Calculus we get $\frac{dy}{dx} = \sin(x^2)$.

Problem Set 5

1. $1 + \cos x = 2\cos^2\left(\frac{x}{2}\right)$, so the integral becomes:

$$\int \frac{1}{2}\sec^2\left(\frac{x}{2}\right) dx = \tan\frac{x}{2} + c.$$

Comment The so-called *double-angle formulae* are often used in integrals of this kind but you have to force yourself to think of the x as 2θ in order to apply them so that $\theta = \frac{x}{2}$. The hint that representation as a square may be useful also comes from the observation that $1 + \cos x \geq 0$ for all x.

2. Put $u = \ln x \Rightarrow du = \frac{dx}{x}$, so the integral becomes:

$$\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln|u| + c = \ln|\ln x| + c.$$

3. $x^2 + 6x + 18 = (x+3)^2 + 3^2$. Put $u = x + 3, du = dx$; our integral becomes:

$$\int \frac{du}{u^2 + 3^2} = \frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + c = \frac{1}{3}\tan^{-1}\left(\frac{x+3}{3}\right) + c.$$

4. Write $I = \int e^x \cos x\, dx$. Put $u = e^x$, $dv = \cos x\, dx$, so that $du = e^x\, dx$, $v = \sin x$. This yields:

$$I = e^x \sin x - \int e^x \sin x\, dx.$$

Call this new integral in the previous expression J. Integrating J by parts: $u = e^x$, $dv = \sin x\, dx \Rightarrow du = e^x\, dx$,$v = -\cos x\, dx$. Hence

$$J = -e^x \cos x + \int e^x \cos x\, dx = -e^x \cos x + I.$$

Hence our expression for I can be re-written as

$$I = e^x \sin x - (I - e^x \cos x) \Rightarrow 2I = e^x \sin x + e^x \cos x \text{ so that}$$

$$I = \frac{1}{2}e^x(\sin x + \cos x) + c.$$

5. $a^{-2x} = e^{(-2\ln a)x}$ and so

$$I = \int_0^1 e^{(-2\ln a)x}\,dx = -\frac{1}{2\ln a}\Big[e^{(-2\ln a)x}\Big]_0^1 =$$

$$-\frac{1}{2\ln a}\Big[e^{-2\ln a} - e^0\Big] = -\frac{1}{2\ln a}\big(a^{-2} - 1\big)$$

$$= \frac{a^2-1}{2a^2\ln a}.$$

6. Put $u = \sin^{-1} x$, $dv = dx$, so that the integral I becomes

$$x\sin^{-1} x + \frac{1}{2}\int \frac{-2x\,dx}{\sqrt{1-x^2}} = x\sin^{-1} x + \frac{1}{2}\int t^{-\frac{1}{2}}\,dt \;\; (\text{where } t = 1-x^2)$$

$$= x\sin^{-1} x + t^{\frac{1}{2}} + c;$$

Therefore a primitive is given by

$$\underline{x\sin^{-1} x + \sqrt{1-x^2}}.$$

7. Put $du = \cos x\,dx$ so that our integral becomes:

$$\int (\sin^2 x)^2 \cos^4 x \cos x\,dx = \int (1-u^2)^2 u^4\,du = \int (u^4 - 2u^6 + u^8)du$$

$$= \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + c.$$

8. $\cos\theta = 1 - 2\sin^2\theta$. Hence $\sin^2 2x = \frac{1}{2}(1-\cos 4x)$ and so the integral becomes:

$$\frac{1}{2}\int (1-\cos 4x)\,dx = \frac{1}{2}\Big(x - \frac{\sin 4x}{4}\Big) + c.$$

9.

$$V = \int_1^\infty \pi y^2\,dx = \pi\int_1^\infty \frac{dx}{x^2}$$

$$= \pi\Big[-\frac{1}{x}\Big]_1^\infty = \pi[0-(-1)] = \pi\,\text{units}^3.$$

Comment Strange as it may seem, the surface area of this object is infinite. That is to say there is an infinite surface to paint but only π cubic units of paint can fit inside! When this was first pointed out by Torecelli around 1640 it caused must consternation.

10. Put $x = 2\sin\theta$ to obtain $dx = 2\cos\theta d\theta$ so that our integral I becomes:

$$I = \int \frac{2\cos\theta d\theta}{4\sin^2\theta\sqrt{4-4\sin^2\theta}} = \int \frac{2\cos\theta d\theta}{4\sin^2\theta\cdot 2\cos\theta}$$

$$= \frac{1}{4}\int cosec^2\theta d\theta = -\frac{1}{4}\cot\theta + c:$$

using the right-angled triangle with angle θ with $\sin\theta = \frac{x}{2}$ we see that the other short side of the triangle has length $\sqrt{2^2 - x^2}$ so that $\cot\theta = \frac{\sqrt{4-x^2}}{x}$. Hence we conclude that

$$I = -\frac{\sqrt{4-x^2}}{4x} + c.$$

Problem Set 6

1.

$$\cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4} = \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} = 1.$$

2.

$$\text{RHS}= \frac{(e^x+e^{-x})(e^y+e^{-y})\pm(e^x-e^{-x})(e^y-e^{-y})}{4}$$

$$= \frac{(e^{x+y} + e^{x-y} + e^{y-x} + e^{-x-y}) \pm (e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y})}{2}$$

$$= \frac{2e^{x\pm y} + 2e^{-(x\pm y)}}{4} = \cosh(x \pm y).$$

3

$$\text{RHS}= \frac{(e^x-e^{-x})(e^y+e^{-y})\pm(e^x+e^{-x})(e^y-e^{-y})}{4}$$

$$= \frac{(e^{x+y} + e^{x-y} - e^{y-x} - e^{-x-y}) \pm (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})}{2}$$

$$= \frac{2e^{x\pm y} - 2e^{-(x\pm y)}}{4} = \sinh(x \pm y).$$

4. Using Questions 2 and 3 we obtain:

$$\tanh(x \pm y) = \frac{\sinh(x \pm y)}{\cosh(x \pm y)} = \frac{\sinh x \cosh y \pm \cosh x \sinh y}{\cosh x \cosh y \pm \sinh x \sinh y}$$

and upon all terms by $\cosh x \cosh y$ we find that

$$\frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}.$$

5. Put $y = x$ in the identities of Questions 2 and 3 respectively and using that of Question 1 gives

$$\cosh 2x = \cosh^2 x + \sinh^2 x = \cosh^2 x + (\cosh^2 x - 1) = 2\cosh^2 x - 1;$$

$$\sinh 2x = \sinh x \cosh x + \cosh x \sinh x = 2 \sinh x \cosh x.$$

6. Sum the two identities $\sinh(u+v) = \sinh u \cosh v + \cosh u \cosh v$ and $\sinh(u-v) = \sinh u \cosh v - \cosh u \sinh v$

$$2 \sinh u \cosh v = \sinh(u+v) + \sinh(u-v).$$

Writing $u+v=x$ and $u-v=y$ so that $u = \frac{x+y}{2}$ and $v = \frac{x-y}{2}$ we obtain from this that

$$\sinh x + \sinh y = 2 \sinh \frac{x+y}{2} \cosh \frac{x-y}{2}.$$

.

7. Re-working the identity of Question 5 we have that $\cosh 2x = 1 + 2\sinh^2 x$ so that $\cosh x = 1 + 2\sinh^2(\frac{x}{2})$ and hence $\sinh^2(\frac{x}{2}) = \frac{1}{2}(\cosh x - 1)$. Since $\sinh x$ is non-negative exactly when $x \geq 0$ we conclude that

$$\sinh\left(\frac{x}{2}\right) = \operatorname{sgn}(x)\sqrt{\tfrac{\cosh x - 1}{2}}.$$

8.

$$y = \sinh^{-1} x \Rightarrow x = \sinh y = \frac{e^y - e^{-y}}{2}$$
$$\Rightarrow u - \frac{1}{u} = 2x, \text{ where } u = e^y,$$
$$\Rightarrow u^2 - 2xu - 1 = 0 \Rightarrow u = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}.$$

Now $u = e^y > 0$, and so only the positive branch is relevant, giving us

$$e^y = x + \sqrt{x^2+1} \Rightarrow y = \ln(x + \sqrt{x^2+1}).$$

Note that the argument of the log function is strictly positive so that this expression is valid as x ranges over the entire real line. We therefore get our expression for the inverse of hyperbolic sine to be:

$$y = \ln\left(x + \sqrt{1+x^2}\right) \ (x \in \mathbb{R}).$$

$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1}).$$

9.

$$y = \cosh^{-1} x \Rightarrow x = \cosh y = \frac{e^y + e^{-y}}{2}$$
$$\Rightarrow u + \frac{1}{u} = 2x, \text{ where } u = e^y,$$
$$\Rightarrow u^2 - 2xu + 1 = 0 \Rightarrow u = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}.$$

Since $\cosh x \geq 1$ we take the positive branch to obtain

$$e^y = x + \sqrt{x^2-1} \Rightarrow y = \ln(x + \sqrt{x^2-1}),\ x \geq 1.$$

10. Put

$$x = \tanh y = \frac{\sinh y}{\cosh y} = \frac{e^y - e^{-y}}{e^y + e^{-y}}.$$

Note that since $|\sinh y| < |\cosh y| \,\forall\, y$ it follows that $-1 < x < 1$. Putting $u = e^y$ then gives a quadratic in u to solve in terms of x:

$$x = \frac{u - u^{-1}}{u + u^{-1}} = \frac{u^2 - 1}{u^2 + 1} \Rightarrow x(u^2+1) - (u^2 - 1) = 0$$

$$\Rightarrow (x-1)u^2 + (x+1) = 0 \Rightarrow u^2 = \frac{-(x+1)}{x-1} = \frac{1+x}{1-x}$$

$$\Rightarrow u = e^y = \sqrt{\frac{1+x}{1-x}} \Rightarrow y = \tanh^{-1} x = \frac{1}{2}\ln\Big(\frac{1+x}{1-x}\Big).$$

Problem Set 7

1.

$$(\cosh x)' = \frac{1}{2}(e^x + e^{-x})' = \frac{1}{2}(e^x - e^{-x}) = \sinh x;$$

$$(\sinh x)' = \frac{1}{2}(e^x - e^{-x})' = \frac{1}{2}(e^x + e^{-x}) = \cosh x.$$

2. Making use of Question 1 and Set 40 Question 1 we get:

$$(\tanh x)' = (\frac{\sinh x}{\cosh x})' = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x},$$

$$\therefore (\tanh x)' = \text{sech}^2 x.$$

3.

$$1 - \tanh^2 x = 1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$$

$$= \frac{1}{\cosh^2 x} = \text{sech}^2 x.$$

4. Put $x = a\sinh t$ so that $dx = a\cosh t$ and so

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \int \frac{a\cosh t\, dt}{\sqrt{a^2 + a^2\sinh^2 t}} = \int \frac{a\cosh t\, dt}{a\sqrt{1 + \sinh^2 t}}$$

$$= \int \frac{a\cosh t\, dt}{a\cosh t} = \int dt = t = \sinh^{-1}\big(\frac{x}{a}\big) + c.$$

5. Put $x = a\cosh t$ with $t \geq 0$ so that $dx = a\sinh t$ and so

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a\sinh t\, dt}{\sqrt{a^2\sinh^2 t - a^2}} = \int \frac{a\sinh t\, dt}{a\sqrt{\cosh^2 t - 1}}$$

$$= \int \frac{a \sinh t \, dt}{a \sinh t} = \int dt = t = \cosh^{-1}\left(\frac{x}{a}\right) + c.$$

Comment: since we have $x^2 \geq a^2$ and $t \geq 0$ here we take $\sinh t$ and not $-\sinh t$ when extracting the root in the denominator.

6. This time put $x = a \tanh x$ with $-1 < x < 1$, which is the range of $\tanh x$. Then $dx = a\text{sech}^2 t dt$and so, making use of Question 3 we obtain

$$\int \frac{dx}{a^2 - x^2} = \int \frac{a\text{sech}^2 t \, dt}{a^2 - a^2 \tanh^2 x} = \int \frac{a\text{sech}^2 t \, dt}{a^2(1 - \tanh^2 x)} = \int \frac{a\text{sech}^2 t \, dt}{a^2 \text{sech}^2 t}$$

$$= \frac{1}{a} \int dt = \frac{t}{a} = \frac{1}{a} \tanh^{-1}\left(\frac{x}{a}\right) + c.$$

7.

$$\sinh x = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}\right)$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(1 + (-1)^{n+1}) x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

$$\cosh x = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{(1 + (-1)^n) x^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

8. Consider the equations $e^{ix} = \cos x + i \sin x$ and $e^{-ix} = \cos x - i \sin x$. Adding and subtracting them respectively yields

$$2 \cos x = e^{ix} + e^{-ix} \text{ and } 2i \sin x = e^{ix} - e^{-ix}$$

$$\Rightarrow \cos x = \cosh(ix), \text{ and } \sin x = -i \sinh(ix).$$

Replacing x by ix in both of these then gives

$$\cos(ix) = \cosh(i^2 x) = \cosh(-x) = \cosh x \text{ and}$$

$$\sin(ix) = -i \sinh(i^2 x) = -i(-\sinh x) = i \sinh x. \text{ Hence}$$

$$\sinh x = -i \sin(ix) \text{ and } \cosh x = \cos(ix).$$

9. Using the double-angle formula and Question 8 we obtain:

$$\cos(x + iy) = \cos x \cos(iy) - \sin x \sin(iy)$$

$$\cos x \cosh y - i \sin x \sinh y.$$

10.

$$\sin(x + iy) = \sin x \cos(iy) + \cos x \sin(iy)$$

$$= \sin x \cosh y + i \cos x \sinh y.$$

Problem Set 8

1. By the Chain rule:

$$y' = -\text{cosec}^2(\sin x)\cos x.$$

2. We have $x = \cot y \Rightarrow \frac{dx}{dy} = -\text{cosec}^2 y$. Now $\text{cosec}^2 y = 1 + \cot^2 y = 1 + x^2$. Hence

$$\frac{dy}{dx} = -\frac{1}{1+x^2}.$$

3. We have $\ln y = x \ln x$ so that

$$\frac{y'}{y} = \ln x + \frac{x}{x} \Rightarrow y' = y(1 + \ln x),$$

$$\therefore y' = x^x(1 + \ln x).$$

4. Let us work in units of 100m and call the length of the sides of the final rectangle x and $1 + y$. Then the area $A(x, y) = x(1 + y)$ and we also have the perimeter is 3 so that $2(1+y)+2x = 3 \Rightarrow 1+y+x = \frac{3}{2} \Rightarrow 1+y = \frac{3}{2} - x$. Hence, as a function of x alone we obtain $A(x) = x\left(\frac{3}{2} - x\right)$. The graph of this function is a downward parabola with roots at $x = 0$ and $x = \frac{3}{2}$ so the unconstrained maximum over all non-negative x occurs at the midpoint between the roots, which is $x = \frac{3}{4}$. However, since $y = \frac{1}{2} - x \geq 0$ we have that $x \leq \frac{1}{2}$. Since $A(x)$ increases in x as we pass from $x = 0$ to $x = \frac{3}{4}$, the maximum value of $A(x)$ given the constraint of the question occurs at $x = \frac{1}{2}$ in which case $y = 0$. The dimensions of the rectangle with the maximum area subject to the constraint in the question is that 50m by 100m (with an area of 5000m^2).

5. Place the origin of our x-axis at the position of the candle of luminosity a so the second candle is at ordinate d on the x-axis. By the Inverse square law, the amount of light from a source diminishes in proportion to the inverse square of the distance of separation. Working in suitable units therefore, if we stand at position x, the amount of light reaching us from the respective sources will be $\frac{a}{x^2}$ and $\frac{b}{(d-x)^2}$. We wish therefore to minimize the sum of these:

$$y = \frac{a}{x^2} + \frac{b}{(d-x)^2}$$

$$\Rightarrow \frac{dy}{dx} = -2ax^{-3} - 2b(d-x)^{-3} \cdot (-1) = \frac{2b}{(d-x)^3} - \frac{2a}{x^3}.$$

Putting this derivative equal to zero then yields the equation:

$$\frac{a}{x^3} = \frac{b}{(d-x)^3} \Rightarrow \frac{(d-x)^3}{x^3} = \left(\frac{d-x}{x}\right)^3 = \frac{b}{a}$$

$$\Rightarrow \frac{d-x}{x} = \left(\frac{b}{a}\right)^{\frac{1}{3}} \text{ and so } \frac{d}{x} = 1 + \left(\frac{b}{a}\right)^{\frac{1}{3}} \text{ and } \frac{x}{d} = \left(1 + \left(\frac{b}{a}\right)^{\frac{1}{3}}\right)^{-1}$$

$$\therefore x = d\Big(1 + \Big(\frac{b}{a}\Big)^{\frac{1}{3}}\Big)^{-1}.$$

You are within your rights, on physical grounds, to claim that this turning point must represent the unique minimum of the function in that range. We can however check that our critical value is a minimum through use of the *Second derivative test*: we just need to verify that the second derivative of the brightness function $y(x)$ is positive at the critical point. Upon differentiating $y' = \frac{2b}{(d-x)^3} - \frac{2a}{x^3}$ we find that

$$y'' = \frac{6b}{(d-x)^4} + \frac{6a}{x^4} > 0$$

for all values of x, so there is no need to substitute the critical value of x into this expression for y'' as we already know the answer will be positive.

As is often the case you can check whether your result gives the right answer in simple cases where the outcome is clear by inspection. Here we note that if $b = a$, then the dimmest point must, by symmetry, be the midpoint between the two candles and indeed if we put $b = a$ in the above formula it returns the expected value of $\frac{d}{2}$. The algebraic manipulations above are all natural enough although you need to resist the temptation to expand the cubic term $(d-x)^3$ the first time that you see it as that will not help at all.

6. In general, the volume of a cone is given by the formula $V = \frac{\pi}{3}r^2h$. The water within the cone at time t forms a cone itself with dimensions $r = r(t)$ and $h = h(t)$ say, but by similar triangles we have $\frac{r}{h} = \frac{2}{5} \Rightarrow r = \frac{2h}{5}$ (since we seek $\frac{dh}{dt}$, we wish to express V in terms of h). Hence the volume of water present is given by $V = \frac{\pi}{3}\big(\frac{2h}{5}\big)^2 h = \frac{4\pi}{75}h^3$. Differentiating with respect to t and applying the Chain Rule now gives:

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt} = \frac{4\pi}{25}h^2\cdot\frac{dh}{dt};$$

now we are told that when $h = 4$ we have $\frac{dV}{dt} = -\frac{1}{12}$. Substituting accordingly yields:

$$-\frac{1}{12} = \frac{4\pi}{25}\cdot 4^2\frac{dh}{dt} \Rightarrow \frac{dh}{dt}|_{h=4} = -\frac{25}{12\times 64\pi} = -\frac{25}{768\pi} \approx 1\cdot 036\text{m/min}.$$

7. By the Fundamental theorem of calculus and the Chain rule we obtain upon putting $u = \sqrt{x}$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \sin(u^2)\cdot(\frac{1}{2}x^{-\frac{1}{2}}) = \frac{\sin x}{2\sqrt{x}}.$$

8. We need

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

$$= \sin x\lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x\lim_{h\to 0}\frac{\sin h}{h} = \sin x(0) + \cos x(1) = \cos x.$$

9. Let m be the gradient of a suitable tangent. Since $(2,0)$ lies on the tangent we have that points (x,y) of the tangent satisfy the relation $m = \frac{y-0}{x-2}$ so that the tangent line has an equation of the form $y = m(x-2)$. Let such a line meets the parabola $y = x^2$ at (a, a^2) say. Then since the line is a tangent to the parabola we have $m = y\prime(a) = 2a$. Equating the y co-ordinates of the tangent and the parabola for $x = a$ now gives $a^2 = 2a(a-2) \Rightarrow a = 0$, or $a = 2(a-2) \Rightarrow a = 4$. Hence the two possible values of m are $m = 2 \times 0 = 0$ or $m = 2 \times 4 = 8$ and the corresponding tangent equations are:

$$y = 0 \text{ or } y = 8(x-2) = 8x - 16.$$

10.

$$y = \log_x a = (\log_a x)^{-1}$$

$$\Rightarrow y' = -(\log_a x)^{-2} \cdot \frac{1}{(\ln a)x} = -\frac{1}{(\ln a)x(\log_a x)^2}.$$

Problem Set 9

1. $t = \tan\frac{\theta}{2}$ throughout Problems 1-4.

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = \frac{2}{\sec^2\frac{\theta}{2}} - 1 = \frac{2}{1+t^2} - 1$$

$$\therefore \cos\theta = \frac{1-t^2}{1+t^2}.$$

2.

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2} = \frac{2\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}$$

$$\therefore \sin\theta = \frac{2t}{1+t^2}.$$

3.

$$t = \tan\frac{\theta}{2} \Rightarrow dt = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta = \frac{1+t^2}{2}\,d\theta$$

$$\Rightarrow d\theta = \frac{2\,dt}{1+t^2}.$$

4. Put $t = \tan\frac{\theta}{2}$,

$$\int \frac{d\theta}{\sin\theta} = \int \frac{1+t^2}{2t} \cdot \frac{2\,dt}{1+t^2} = \int \frac{dt}{t} = \ln|t| + c$$

$$\therefore \int cosec\theta \ \mathrm{d}\theta = \ln|\tan\tfrac{\theta}{2}| + c.$$

5.

$$\int \frac{d\theta}{\cos\theta} = \int \frac{1+t^2}{1-t^2}\cdot\frac{2\,dt}{1+t^2} = \int \left(\frac{1}{1-t}+\frac{1}{1+t}\right) dt = \ln(1+t)-\ln(1-t)+c$$

$$\therefore \int \sec\theta\, d\theta = \ln\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}} + c.$$

Comment This is not the neatest solution. In Question 8 on Set 1 we showed that an alternative answer is $\ln\frac{\cos\theta}{1-\sin\theta}$. Another inspired approach is to multiply top and bottom of the integrand by $\sec\theta+\tan\theta$ to get

$$I = \int \frac{\sec\theta+\tan\theta}{\sec^2\theta+\sec\theta\tan\theta}\, d\theta = \ln(\sec\theta+\tan\theta).$$

Of course all these answers must be equal up to an additive constant.

6. Placing the three partial fractions over a common denominator gives:

$$\frac{ax(x-1)+b(x-1)+cx^2}{x^2(x-1)} \equiv \frac{-1}{x^2(x-1)}$$

Put $x=0$ and equate coefficients: $-b=-1 \Rightarrow b=1$. Putting $x=1$ gives $c=-1$. Equating coefficient of x^2 to 0 gives: $a+c=0 \Rightarrow a=-c=-(-1)=1$.

$$\therefore \frac{1}{x^2(1-x)} = \frac{1}{x}+\frac{1}{x^2}-\frac{1}{x-1}.$$

Comment The Cover Up method can be used to find b and c (but not a): covering the term in x^2 and putting $x=0$ gives $b=\frac{-1}{0-1}=1$, while covering up the term in $x-1$ and putting $x=1$ gives $c=\frac{-1}{1^2}=-1$.

We now apply the given decomposition:

$$\int \frac{dx}{x^2(1-x)} = \int \frac{dx}{x} - \int \frac{dx}{x-1} + \int \frac{dx}{x^2} = \ln|x| - \ln|x-1| + \Big(-\frac{1}{x}\Big) + c$$

$$= \ln\Big|\frac{x}{x-1}\Big| - \frac{1}{x} + c.$$

7. We may calculate the volume of revolution 'by washers':

$$V = \int_0^1 \pi(R^2-r^2)\, dy = \pi\int_0^1 (y-y^2)\, dy$$

$$= \pi\Big[\frac{y^2}{2}-\frac{y^3}{3}\Big]_{y=0}^{y=1} = \pi\Big(\big(\frac{1}{2}-\frac{1}{3}\big)-(0-0)\Big) = \frac{\pi}{6}.$$

Or we may calculate this by the method of 'cylindrical shells':

$$V = \int_0^1 2\pi x f(x)\, dx = 2\pi\int_0^1 x(x-x^2)\, dx$$

$$= 2\pi \int_0^1 (x^2 - x^3)\, dx = 2\pi \Big[\frac{x^3}{3} - \frac{x^4}{4}\Big]_{x=0}^{x=1}$$

$$= 2\pi\big((\frac{1}{3} - \frac{1}{4}) - (0-0)\big) = \frac{\pi}{6}.$$

8.

$$\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \tan x - x + c.$$

9.

$$\sin mx \cos nx = \frac{1}{2} \sin\big((m+n)x\big) + \frac{1}{2} \sin\big((m-n)x\big)$$

and so our integral becomes:

$$\frac{1}{2}\int_0^{2\pi} \big(\sin\big((m+n)x\big) + \sin\big((m-n)x\big)\big)\, dx$$

$$= \frac{1}{2}\Big[\frac{-\cos\big((m+n)x\big)}{m+n} - \frac{\cos\big((m-n)x\big)}{m-n}\Big]_0^{2\pi}$$

$$= \frac{1}{2}\Big[\big[-\frac{\cos\big(2(m+n)\pi\big)}{m+n} - \frac{\cos\big(2(m-n)\pi\big)}{m-n}\big] - \big[-\frac{1}{m+n} - \frac{1}{m-n}\big]\Big]$$

$$= \frac{1}{2}\Big[-\frac{1}{m+n} - \frac{-1}{m-n} + \frac{1}{m+n} + \frac{1}{m-n}\Big] = 0.$$

Comment That this integral is zero is used constantly in the calculation of coefficients in *Fourier series*, which is the fundamental tool for expressing a periodic function in series form (see Set ??).

10. As with any of the inverse transcendental functions, we integrate by parts: put $u = \arctan x$, $dv = dx$ so that $du = \frac{dx}{1+x^2}$ and $v = x$ to obtain:

$$x \arctan x - \int \frac{x\, dx}{1+x^2},$$ and substituting for $1 + x^2$in this intergral gives:

$$x \arctan x - \frac{1}{2}\ln(1+x^2) + c.$$

Problem Set 10

1. Let $I = \int x^2 e^{-x} dx$. Integrate by parts, putting $u = x^2, dv = e^{-x}dx$ so that $du = 2xdx$, $v = -e^{-x}$. Hence

$$I = -x^2 e^{-x} + 2\int xe^{-x}\, dx.$$

Call this new integral J and integrate by parts again with $u = x$, $dv = e^{-x}dx$ so that $du = dx$ and $v = -e^{-x}$. We obtain

$$J = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x}.$$

Hence $I = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + c$ and therefore:

$$I = -e^{-x}(x^2 + 2x + 2) + c.$$

2. Integrating by parts twice: put $u = e^x$, $dv = \sin x\,dx \Rightarrow v = -\cos x$ so that

$$I = -e^x\cos x + \int e^x \cos x\,dx.$$

Integrating the new integral by parts in the same fashion gives

$$I = -e^x\cos x + (e^x \sin x - \int e^x \sin x\,dx) \Rightarrow 2I = e^x\sin x - e^x\cos x,$$

$$\therefore I = \frac{1}{2}e^x(\sin x - \cos x) + c.$$

Our integral is $I = \int e^{-x}\sin x\,dx$. Putting $u = -x$ transforms I into $-\int e^u \sin(-u)\,du = \int e^u \sin u\,du$. By above this gives

$$I = \frac{1}{2}e^u(\sin u - \cos u) = \frac{1}{2}e^{-x}(\sin(-x) - \cos(-x))$$

$$\Rightarrow \int \frac{\sin x}{e^x}\,dx = c - \frac{\sin x + \cos x}{2e^x}.$$

3. Let $f(x) = \sin^{2n+1} x$. Then

$$f(-x) = (\sin(-x))^{2n+1} = (-\sin x)^{2n+1} = (-1)^{2n+1}(\sin^{2n+1} x) = -\sin^{2n+1} x = -f(x).$$

Hence $f(x)$ is odd and so $\int_{-1}^{1} f(x)\,dx = 0$.

4. $\frac{x^2}{1+x^2} = \frac{(1+x^2)-1}{1+x^2} = 1 - \frac{1}{1+x^2}$. Hence our integral becomes:

$$\int \big(1 - \frac{1}{1+x^2}\big)\,dx = x - \arctan x + c.$$

5.

$$I = \lim_{l\to 1^-}\int_0^l \frac{dx}{\sqrt{1-x}} = \lim_{l\to 1^-}\big[-2\sqrt{1-x}\big]_0^l$$
$$= \lim_{l\to 1^-}\big[-2\sqrt{1-l} + 2\big] = 0 + 2 = 2.$$

6. Putting $\sin x + \cos x = r\cos(x - \alpha)$ gives $r = \sqrt{2}$, $\alpha = \frac{\pi}{4}$. Using that $\int \sec x\,dx = \ln|\sec x + \tan x|$ then gives

$$\int \frac{dx}{\sin x + \cos x} = \frac{\sqrt{2}}{2}\int \sec(x - \frac{\pi}{4})\,dx = \frac{\sqrt{2}}{2}\ln|\sec(x - \frac{\pi}{4}) + \tan(x - \frac{\pi}{4})| + c.$$

7. Putting $u = x + 1$ the integral I becomes

$$\int \frac{\sin(u-1)}{\cos u}\, du = \int \frac{\sin u \cos 1 - \cos u \sin 1}{\cos u}\, du =$$

$$(\cos 1)\int \tan u\, du - (\sin 1)\int du = (\cos 1)\ln|\sec u| - (\sin 1)u + c =$$

$$(\cos 1)\ln|\sec(x+1)| - (\sin 1)x + c,$$

where we have absorbed the term $-\sin 1$ into the additive constant.

8. With $x = a\sin t$ $(-\frac{\pi}{2} \leq t \leq \frac{\pi}{2})$ (range determined by the range of x, which is $-a \leq x \leq a$) we have $dx = a\cos t dt$ and $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2 t} = \sqrt{a^2\cos^2 t} = a\cos t$. Hence

$$I = \int (a\cos t)^2\, dt = \frac{a^2}{2}\int (1 + \cos 2t)\, dt = \frac{a^2}{2}\big(t + \frac{1}{2}\sin 2t\big) + c.$$

Now $t = \arcsin\frac{x}{a}$ and so $\frac{1}{2}\sin 2t = \sin t\cos t = \frac{x}{a}\sqrt{1 - \frac{x^2}{a^2}} = \frac{x}{a^2}\sqrt{a^2 - x^2}$ so in terms of our original variable x we have:

$$I = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + c.$$

9. With $x = a\cos t$ $(0 \leq t \leq \pi)$ we have $dx = -a\sin t\, dt$ and $\sqrt{a^2 - a^2\cos^2 t} = \sqrt{a^2\sin^2 t} = a\sin t$. Hence

$$I = -\int (a\sin t)^2\, dt = \frac{a^2}{2}\int (\cos 2t - 1)\, dt = \frac{a^2}{2}\big(\frac{1}{2}\sin 2t - t\big) + c =$$

$$-\frac{a^2}{2}\arccos\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + c.$$

10. The answers to 8 & 9 are consistent as the expressions differ by a fixed constant, that being:

$$\frac{a^2}{2}\big(\arcsin\frac{x}{a} + \arccos\frac{x}{a}\big) = \frac{a^2}{2}\frac{\pi}{2} = \frac{\pi a^2}{4}.$$