Mathematics 208 Classical Mechanics

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This module on classical mechanics follows on and presumes the content of MA108 *Mechanics*. The central theme is that of rotating bodies and so we see in Sets 1 and 2 the focus is on problems involving the *centre of mass* and the *moment of inertia* of a system of particles and of a rigid body in two and three dimensions, including the *Perpendicular* and *Parallel axis theorems.* Sets 3 and 4 are based on energy and work considerations related to moments of inertia and feature standard problems involving the motion of masses on inclined planes, over pulley systems and the *torque* that results within systems subjected to external forces. Set 5 features questions involving both the *static* and *kinetic coefficient of friction* when forces are in play on objects moving over rough surfaces.

In the latter half of the module we introduce some new techniques apart from our standard approaches of the use of Newton's laws and Conservation of energy. Set 6 calls upon the technique of *Virtual work* to resolve forces on systems in equilibrium. Just as force is the rate of change of linear momentum, *torque* is the time derivative of *angular momentum* and these concepts are the work of Set 7. The *Euler-Lagrange equation* is introduced in Set 8 as an alternative to the Newtonian scheme in mechanics questions. Sets 9 and 10 have as their subject rotating frames of reference. *Coriolis forces* are explored in Set 9 while in Set 10 we return to the topic of *central forces.*

Problem Set 1: Centre of Mass and Moment of Inertia

The *centre of mass* of a body that consists of point masses m_i with position vectors $\boldsymbol{r}_i$ is $\frac{\sum_i m_i \boldsymbol{r}_i}{\sum_p m_i}$. The following facts are framed for continuous bodies and so come in integral notation. Discrete counterparts are expressed in summation notation. The *(first) moment* of a lamina R with density $\delta(x,y)$ about the x-axis is $M_x = \int\int_R y\delta(x,y)\,dA$; the *centre of mass* lies at $(\overline{x},\overline{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$, where $M = \int\int_R \delta(x,y)\,dA$.

For a uniform lamina defined as the area above the interval $[a,b]$ and below the graph of $y = f(x)$ the required coordinates are $\overline{x} = \frac{\int_a^b xy\,dx}{\int_a^b y\,dx}$, $\overline{y} = \frac{\int_a^b \frac{1}{2}y^2\,dx}{\int_a^b y\,dx}$.

The *second moment*, also known as the *Moment of inertia* about the x-axis is $I_x = \int\int_R y^2\delta(x,y)\,dA$ and about the origin is $I_0 = I_x + I_y$. The *radius of gyration* about the x-axis is $R_x = \sqrt{\frac{I_x}{M}}$ and about the origin $R_0 = \sqrt{\frac{I_0}{M}}$. The first moments for a mass M in a region R of 3-space about coordinate planes have the form $M_{yz} = \int\int\int_R x\delta\,dV$; *Centre of mass*: $(\overline{x},\overline{y},\overline{z}) = \frac{1}{M}(M_{yz}, M_{zx}, M_{xy})$. $I_x = \int\int\int_R (y^2+z^2)\delta\,dV$.

1. Find $\overline{x}$ for a discrete set of point masses $\{m_k\}$ each at position x_k with respect to the origin O by equating the first moment about $\overline{x}$ to zero.

2. Find the centre of mass of four points occupying the corners of a square of side length a, the containing masses of which, listed anti-clockwise are $2, 3, 6, 5$. Take the origin at the point of mass 2 and the x- and y-axes in the direction of the masses 3 and 6.

3(a) Consider an object to be made up of two disjoint sets of point masses A and B with x-coordinates of the respective centres of mass $\overline{x}_A$ and $\overline{x}_B$. Let M_A and M_B be their respective masses. Show that

$$\overline{x} = \frac{M_A\overline{x}_A + M_B\overline{x}_B}{M},$$

where M is the total mass of the body and $\overline{x}$ is the x-coordinate of its centre of mass.

(b) Find the centre of mass of an L shape with vertices positioned at

$$(0,11), ((0,0), (5,0), (5,1), (1,1), (1,11).$$

4. Find the centre of mass of a right triangle with shorter sides of lengths a and b using the double integral approach.

5. Solve Question 4 use the alternative single-integral approach.

6. Find the centre of mass of a semi-circular uniform lamina of radius a.

7. Find the centre of mass of a thin plate of density δ defined by the parabola $y = 4 - x^2$ and the x-axis.

8. Repeat Question 7 with the density function given by $\delta(x, y) = 2x^2$.

Consider the triangular lamina bounded by the x-axis, the lines $x = 1$ and $y = 2x$ with density $\delta(x, y) = 6x + 6y + 6$.

9. Find the plate's mass M, its first moments M_x and M_y, and its centre of mass.

10. Find the moments of inertia I_x and I_y and radii of gyration about the coordinate axes.

Problem Set 2: Moment of inertia additive theorems

1. Find the *centroid* (centre of mass of a uniform lamina) of the region in the first quadrant bounded above by the line $y = x$ and below by the curve $y = x^2$.

2. Find I_x for the regular solid cuboid of unit density with respective dimensions a, b, and c taking the origin as the centre of the block.

3. Find the centre of mass of the solid of unit density bounded by the disc $x^2 + y^2 \leq 4$ in the plane $z = 0$ and bounded above by the parabaloid $z = 4 - x^2 - y^2$.

4. *Perpendicular axis theorem* Consider a rigid object that lies entirely within the xy-plane and let I_x, I_y, and I_z denote the respective moments of inertia of the body around each of the coordinate axes. Show that

$$I_z = I_x + I_y.$$

5. *Parallel axis theorem* Show that if a body of mass m rotates about an axis z' displaced a distance d from an axis z through the centre of gravity of the body then its moment of inertia is increased by md^2.

Find the moments of inertia of these objects about the given axis in terms of their mass m and other relevant parameters.

6(a) A thin rod with cross-section s, density ρ, length l about an axis through its centre of mass and perpendicular to its length.
(b) Use the parallel axis theorem to find the moment of inertia of the rod about an axis perpendicular to the rod and through an endpoint of the rod.

7. Consider a cylinder of radius R, length l and mass m. Find I about an axis:
(a) through its centre and parallel to its length;
(b) through a diameter of one of the circular faces;
(c) through the centre of the cylinder perpendicular to its length.

8. Find I for a uniform rectangular sheet in the xy-plane with sides a and b respectively and with centre at the origin:
(a) along the x-axis;
(b) through the edge parallel to the x-axis.

9. Continue Question 8 with axis:
(a) through the centre perpendicular to the plane;
(b) through a corner with axis perpendicular to the plane;
(c) through the centre of an edge parallel to the x-axis and perpendicular to the plane.

10. A solid cone of height h and base radius R with axis through its apex and perpendicular to its base.

Problem Set 3: Moments and Energy

1(a) Find the moment of inertia of a uniform sphere of density ρ and radius R about an axis through its centre.

(b) Use the result of part (a) to write down the moment of inertia of a partly hollow sphere of mass m, inner radius R_2 and radius $R = R_1$.

2. Find the moment of inertia of a thin spherical shell of mass m and radius R by writing $R = R_2 + r$ in Question 1 and take the limit as $r \to 0$.

The energy E of a body rotating about an axis through its centre of mass with angular velocity ω is given by $E = \frac{1}{2}I\omega^2$, where I is the moment of inertia about the given axis.

3(a) Explain the previous statement for a system of discrete particles.

(b) By using integral notation, show this for a solid body.

4. Find the velocity of a sphere of uniform density with mass m and radius R after it has rolled down an inclined plane a vertical distance of h.

5. Repeat Question 4 with a thin spherical shell of the same mass and radius.

6(a) Repeat Question 4 for the cylinder of radius R, length l and mass m.

(b) What would be the result of a rolling race between the uniform sphere, spherical shell, and cylinder, all of the same mass and radius?

7. A thin hoop of radius R rolls down a hill with total drop of h. How fast will the hoop be moving when it reaches the bottom?

8. Show that in polar co-ordinates the centroid of a uniform region R are:

$$\overline{x} = \frac{1}{\text{area of } R}\int\int_R r^2 \cos\theta \, d\theta, \ \overline{y} = \frac{1}{\text{area of } R}\int\int_R r^2 \sin\theta \, d\theta.$$

9. Find the centroid of the region enclosed by the *cardiod* $r = a(1 + \sin\theta)$.

10. As in Question 9 but now work on the petal of the rose, $r = \sin 2\theta$ that lies in the first quadrant.

Problem Set 4: Torque and work

1. A torque of 50Nm is applied to a wheel causing it to rotate 60 times in 12 seconds. Find the work done by the torque and the mean power produced by the torque over the time period.

2. A disc pulley of mass M is mounted on a vertical wall with centre at height h above the ground. A mass m hangs from a light string wrapped around the pulley. The mass is allowed to fall from height h.
(a) Use the angular acceleration formula $\tau = I\alpha$, where α is the angular acceleration, to find the tension T in the string as the mass falls in terms of a, the acceleration of the falling mass.
(b) Find the acceleration of the falling mass, showing it is constant.
(c) At what speed does the falling mass strike the ground?

3. Solve part (c) of Question 2 directly using Conservation of energy considerations.

4. Two masses, m_1 and m_2 are joined by a light string that passes over a disc pulley of mass M and radius R. The mass m_1 lies on a smooth horizontal surface at the same level as the top of the pulley. The second mass, which hangs down from the pulley, is then released and falls under gravity.
(a) Find the tensions T_1 and T_2 between the respective masses and the pulley.
(b) Find the common acceleration a of the two masses.

5(a) Repeat Question 4 but this time allow for a coefficient of friction μ retarding the motion of m_1 and with the surface under m_1 being a plane inclined at an angle θ to the horizontal.
(b) Find the relative values of m_2 and m_1 under which the system will be in equilibrium.

6. A solid cylinder of radius R and mass m rolls down a plane inclined at angle θ where the coefficient of friction is μ.
(a) Use Newton's law to find the acceleration of the cylinder in terms of θ and μ;
(b) Use the equation $\tau = I\alpha$ to express μ in terms of θ and a;
(c) Hence find a as a function of θ.
(d) Find the maximum value of θ which allows the cyclinder to roll (without any sliding) down the slope.

7. A uniform beam of length L is attached to a rope at a distance a from the left hand end of the beam. A mass m hangs from the right hand end of the beam, which is then in stable equilibrium. Find the mass of the beam.

8. A yo-yo can be considered as a disc of radius R and mass m. However the string acts on an inner cylinder of radius $r \leq R$.
(a) Find the acceleration of the freely descending yo-yo.

(b) What value of r yields the maximum acceleration possible, and what is the value of that acceleration?

9. A horizontal plank of length L is supported at its endpoints, A and B. A mass m is placed at a distance a from A. By considering the the resulting torques at A and B due to the weight of m, find the normal reaction forces R_A and R_B at A and B respectively by considering the net torque at each of A and B.

10. A piston moves in a vertical shaft. The shaft is a horizontal distance d from a point A where a beam is hinged. The beam has a movable sleeve on it, which is attached to the piston so that motion of the beam as it pivots at A allows the sleeve to push the piston up and down in the shaft. Let the angle between the beam and the vertical be denoted by θ.

A moment of M is applied at A acting to lift the beam while a force P acts directly down on the piston. Show that to maintain equilibrium requires that the value of the moment satisfies:

$$M = \frac{Pd}{\sin^2 \theta}.$$

Problem Set 5: Coefficient of friction problems

1. A block weighing 100N is pushed along a surface. If it takes 40N to get the block moving and 20N to keep the block moving at a constant velocity, what are the *coefficients of friction* μ_s (static) and μ_k *(kinetic)* μ_k?

2. A car of mass 1200kg is travelling at 20m/sec when it brakes and skids to a stop. The coefficient of friction between the tyres and road is $0 \cdot 8$. Find:
(a) the deceleration of the car;
(b) the distance travelled by the car before coming to rest;
(c) the time taken for all this to happen.

3. A body of mass m rests on a horizontal surface whose coefficient of friction is $\mu_s = \mu_k = \mu$. A force F is applied at an angle of θ above the horizontal away from the surface (upward). Find:
(a) the acceleration a of mass;
(b) the maximum force F that may be applied so that the mass will remain at rest.

4. Two masses m and M are connected by a light string over a pulley. Mass M is on a level surface with coefficient of friction $\mu_s = \mu_k = \mu$ while m is free to fall apart from the support of the tension in the string. Find:
(a) the common acceleration of the masses, stating under what conditions the system is set in motion.
(b) the coefficient of friction μ if $M = 5$kg, $m = 1$kg and the speed of the masses is constant.

5. Repeat Question 4(a) but now with the mass M being pulled up a slope at an angle of θ to the horizontal:
(a) in the case of upward acceleration of M;
(b) in the case of downward acceleration of M.

6. For the system of Question 5, determine the acceleration and the motion direction if $M = 4$kg, $m = 2$kg, $\theta = 45°$ and the coefficient of friction coefficient is $0 \cdot 1$.

7. A light string with one end fixed to a horizontal plane supports a movable pulley (below the plane), from which hangs a mass m_1. The string then passes over a fixed pulley (above the height of the first) and the other end of the string hangs over the second pulley, supporting a mass m_2.
(a) Write down equations of motion for each mass;
(b) Write down an equation that relates to the two accelerations a_1 and a_2 of the respective masses.
(c) Solve these three equations for each of the unknowns a_1, a_2, and T, the tension in the string.

8. (a) Consider a system of two massless pulleys where a fixed pulley P_1 (suspended by a string from the ceiling) supports by a string S_1 a mass m_1 on

the left and on the right supports another pulley P_2. The movable pulley P_2 in turn supports masses m_2 and m_3 on its left and right by a separate string S_2.

Write down five equations for the five unknowns, a_1, a_2, a_3 of accelerations of the masses m_i and tensions T_1 and T_2 in the respective strings S_1 and S_2.

(b) Set up a 3×3 matrix equation in the unknowns a_1, a_2, and T_2.

9(a) Solve the equation system of 8(b) using Cramer's rule.

(b) Give the full solution set for the problem of Question 8.

(c) What is the result if all three masses equal a common value of m?

10. Three masses m_1, m_2, m_3 are joined, in that order by light ropes and the trio of masses is towed by a horizontal force F applied to m_3 along a plane with coefficient of friction $\mu_k = \mu$.

(a) Find the acceleration a of the trio of masses;

(b) Find the tensions T_1 and T_2 in the ropes joining m_1 to m_2 and m_2 to m_3 respectively.

Problem Set 6: Virtual Work

Principle of virtual work The total work done by all the external *active forces* (forces with a non-zero component in the direction of the virtual displacement) and torques on an ideal mechanical system in equilibrium is zero for all virtual displacements consistent with the constraints of the system. This is written as $\delta U = 0$. The sum δU will consist of terms of the form $\overrightarrow{F} \bullet \overrightarrow{\delta r}$, where $\overrightarrow{F}$ is a force acting at a point that undergoes an incremental vector displacement $\overrightarrow{\delta r}$, and moment terms.

1. Solve Question 9 Set 4 using the *Principle of virtual work* by considering the virtual work done by a small upward displacement δy of the platform and equating the sum of the work done by the active forces to 0.

2. Again solve Question 1, this time by considering the virtual work done by a torque acting at point A raising the platform through a small angle $\delta\theta$.

3. Two light struts of equal length L are joined at their endpoints allowing them to pivot freely at this common point P. The other ends of the struts are fixed below the level of the pivot on the horizontal ground at points A and B respectively. The angle between the struts is 2θ. A force F pushes downwards at P.

(a) Let y denote the vertical coordinate of the joint P where F acts and let $x = AB$. Express y and x in terms of θ and L and find the expressions for the virtual displacements δy and δx in terms of $\delta\theta$.

(b) Use the Principle of virtual work to find the force B_X acting at B on the strut fixed with endpoint B.

4. The *mechanical efficiency* of a system is the ratio

$$e = \frac{\text{work output}}{\text{work input}}.$$

Show that (in the absence of friction) for the system of Question 3 the efficiency is 100% by verifying that the ratio $\frac{R\,\delta x}{F\,\delta y} = 1$.

5. Repeat Question 4 but now allow for the a non-zero coefficient of friction μ.

(a) Show that the frictional force acting opposite the horizontal component of F acting through the right hand strut is given by $\frac{1}{2}F\mu$.

(b) Find the magnitude B_X of the resistance force in these circumstances;

(c) Show that in this case

$$e = 1 - \mu \cot\theta.$$

(d) Comment on this result in the case of small θ (so the right hand side gives a negative number).

6. Solve Question 10 of Set 4 as follows:

(a) Express y in terms of θ and the *virtual displacement* δy in terms of $\delta\theta$;

(b) Find the total *virtual work* done by the forces;

(c) Solve for the unknown torque M by equating the virtual work to zero.

7. Two rods AB and BC, each of length l and mass m, are joined and may pivot at their common point B below the horizontal AC. The point C is constrained to move only along a horizontal track while A is fixed at the same level. A force F acts at the midpoint P of BC in the (horizontal) direction of AC. Let the angle between the rods be denoted by θ.

(a) How many degrees of freedom are there for this system?

(b) Let x be the horizontal coordinate with positive direction from A to C. Express x_P, the x-coordinate of the point P where F acts, and the virtual displacement δx, in terms of θ and $\delta\theta$.

(c) Set the virtual work done by the active forces to zero in order to find the value of F that prevents the rods from collapsing together.

8. A spring is fixed at a point A with C the free moving end of the spring. The spring is constrained to move in a vertical slot. A pair of arms, each of length l, have their ends at A and C respectively with the arms attached to each other at their common endpoint B about which they are free to pivot. A force P acts downwards at the join of the arms at B. Let θ denote the angle between the horizontal and the upper arm AB. The unstretched length of the spring is h and the spring constant is k. Let y_C be the vertical coordinate of C measured from A.

(a) Determine the expression for y_C and y_B and find the spring force for equilibrium position of the system;

(b) Express the total virtual work done by the active forces;

(c) Use the Principle of virtual work to determine θ in terms of P, h, k, and l for the system in equilibrium.

9. A mass m is placed on a horizontal platform at height h above the ground. The platform is supported by two struts of length $2b$ both inclined at an angle θ to the horizontal and leaning to the right (so $\theta < 90°$). The struts are free to pivot about their bases so in order to prevent collapse a hydraulic jack supports the right strut at its centre with the base of the jack a distance L from the base point of the right strut. Find the force F exerted by the jack that will maintain the system in equilibrium.

10. A beam AB has length 5m has a force in the direction DC of 20N applied at the point C, which is 3m from A with $\angle DCB = 60°$. A light upright beam EA of length 6m is mounted at A and a force of 8N is applied at E at an angle 45° upward and to the left.

(a) Find the vertical reaction force R_B at B by considering the virtual work that results from a small rotation $\delta\theta$ at A.

(b) Then find R_A from your answer to (a) and Newton's law.

Problem Set 7: Angular momentum

1. Given that the work done in moving a mass m from an initial position i to final position f in the presence of a force vector $\boldsymbol{F}$ is given by

$$W_{i,f} = \int_i^f \boldsymbol{F} \bullet \boldsymbol{dr}$$

Show that

$$= m \int_{i(\boldsymbol{v})}^{f(\boldsymbol{v})} \boldsymbol{v} \bullet d\boldsymbol{v} = \frac{1}{2} m(v_f^2 - v_i^2);$$

where v_i and v_f denote the magnitudes of the initial and final velocities respectively.

2. The *angular momentum* of a moving body and the *torque* (rotational force) acting on a body are respectively given by:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}, \ \boldsymbol{D} = \boldsymbol{r} \times \boldsymbol{F}$$

where $\boldsymbol{p}$ denotes the momentum vector of the body. Show that $\dot{\boldsymbol{L}} = \boldsymbol{D}$.

Question 3 and 5-7 concern the *simple pendulum*: consider a pendulum mounted at a frictionless pivot P consisting of a light rod of length l and a bob of mass m that is swinging freely.

3(a) Working in terms of radial and angular coordinates with origin P and x-axis directed downwards from P show that the torque vector of the system is given by:

$$\boldsymbol{\tau} = -mgl \sin\theta \boldsymbol{k},$$

where θ is the angular displacement of the rod from the vertical.

(b) By means of the equation $\tau = I\alpha$, where α denotes the *angular acceleration*, show that θ satisfies the differential equation:

$$\ddot{\theta} + gl \sin\theta = 0.$$

(c) Derive the same equation using the general formula for acceleration in polar and transverse components:

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}.$$

4(a) Show that the radial and transverse components of velocity and of acceleration come from differentating $x = r\cos\theta$ with respect to time and setting $\theta = 0$ and $\theta = \frac{3\pi}{2}$ respectively. Explain why this happens.

(b) Show that the transverse component of acceleration may also be expressed as:

$$\frac{1}{r}\frac{d}{dt}\big(r^2\dot{\theta}\big).$$

5. (a) For small angles θ we may approximate $\sin\theta$ by θ in the differential equation of Question 3(b). By doing this, find the general solution for $\theta(t)$.

(b) Find the particular solution if the pendulum starts from rest at an intial angle of θ_0.

(c) Find the period T of the pendulum.

6. Use the *vector triple product* and the equation $\boldsymbol{v} = \boldsymbol{r} \times \boldsymbol{\omega}$ to derive the equation $\boldsymbol{L} = mr^2\boldsymbol{\omega}$ from the definition $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ of the angular momentum vector $\boldsymbol{L}$ of a particle, where $\boldsymbol{r}$ and $\boldsymbol{v}$ are respectively the position and velocity vectors of the particle.

7(a) A bob is swung vertically on a light string of length l. Find the minimum value of the angular velocity ω at the bottom of the swing that will ensure the bob swings through a complete circle.

(b) Repeat part (a) with the string replaced by a light rod.

8. A thin uniform rod AB of length $2a$ rotates freely about a horizontal axis through A. The rod is released from the horizontal.

(a) By differentiating the Conservation of energy equation, show that $a\ddot{\theta} = \frac{3}{4}\cos\theta$, where the angle θ is measured from the horizontal.

(b) Find the components of force along and perpendicular to the rod applied at A by the hinge on the rod when the rod has fallen through an angle of θ.

9. A thin uniform rod AB stands on its end A and then topples over. It begins to slip after it has rotated through $30°$. Find the value of μ, the coefficient of friction between the rod and the surface.

10. A door of width $2a$ and mass m swings freely on a smooth vertical hinge until in meets a door stop. How far from the axis of the door should the door stop be placed so that the impulse along the door hinge that results from impact is zero?

Problem Set 8: Euler-Lagrange equation

The quantity L is the difference $L = T - V$ of the kinetic energy T and potential energy V of the system. We express L, the *Lagrangian*, as a function of n *generalised coordinates*, together with their time derivatives and time itself: $L(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t)$. We solve the problem using the fact that L then satisfies the *Euler-Lagrange equation of motion*:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}.$$

1. A tube rotates in the xy-plane with constant angular velocity ω. The tube contains a sphere of mass m.

(a) What is the number of degrees of freedom of the problem?

(b) Write down the Lagrangian for this system;

(c) Using r, the radial coordinate of the sphere as a generalized coordinate, express L in terms of r, ω, and m.

2. Solve the Euler-Lagrange equation for the system of Question 1.

3. A particle moves without friction on a cycloidic path:

$$x(\theta) = a(\theta - \sin\theta),\ y(\theta) = a(\theta + \cos\theta),\ \theta \in [0, 2\pi].$$

(a) Find the Lagrangian of the system;

(b) Find the Euler-Lagrange equation.

4. Show that the Euler-Lagrange equation of Question 3 may be written as:

$$\ddot{\theta} + \frac{1}{2}\cot(\frac{\theta}{2})\dot{\theta}^2 - \frac{g}{2a}\cot(\frac{\theta}{2}) = 0.$$

5. Solve the equation in Question 4 by use of the substitution:$u = \cot(\frac{\theta}{2})$.

6. A pendulum consisting of a light rod of length l and a bob of mass m swings about it support point. Show that the angle between the rod and the vertical is governed by the equation:

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

(a) by Newtonian analysis of forces;

(b) by use of the Euler-Lagrange equation.

7. A bead of mass m is free to slide around a frictionless hoop of radius R, which is itself rotating about the vertical axis with constant angular velocity ω.

(a) Write down two holomonic constraints for this system, showing there is one degree of freedom for the problem.

(b) Using the angle θ between the radius of the hoop to the bead and the z-axis, write $\dot{x}, \dot{y}$, and $\dot{z}$ in terms of θ and R;

(c) Write down the Lagrangian $L = T - V$ for this problem.

8. For the problem of Question 7, determine the Euler-Lagrange equation and thereby express $\ddot{\theta}$ in terms of the generalized co-ordinate θ.

9. A wedge of mass M, height h, and with acute angle with the horizontal of α, is free to slide on a horizontal surface. A mass m is placed at the top of the wedge and slides freely without friction.

(a) Show that this system has two degrees of freedom, which we take to be the distance q the mass has slid down the wedge and the distance x the wedge has moved from its initial position with its right-angled corner at the origin by expressing the coordinates of the small mass (x_m, y_m) in terms of the x-coordinate x_M of the right-hand wall of the wedge, the variables q, and h and α.

(b) Write down the Lagrangian of the system in our generalized coordinates.

10(a) For the problem of Question 9, write down the Euler-Lagrange equation for each of the generalized coordinates;

(b) Hence find the exact value of the acceleration of the wedge, of the sliding mass relative to the wedge, and of the sliding mass relative to the ground.

Problem Set 9: Rotating frames of reference

1. *Coriolis force* A mass m is moving outwards with radial velocity v_r along a radial line from the centre of a spinning disc of angular velocity ω.

(a) What is the angular momentum of the mass when at a distance r from the centre?

(b) Write the torque τ as $\tau = F_c r$ where F_c is the so-called Coriolis force being exerted on the mass in order for this to happen. Use $\tau = \dot{L}$ to find the value of F_c.

2. Let A be an observer at the centre of a spinning caruousel of radius r as in Question 1 and B be standing on the ground outside. Let v_A be velocity of a point P moving around the rim with respect to A.

(a) What is v_B, the velocity of P with respect to B?

(b) What is force acting on P in terms of v_B?

(c) What is the force on P in terms of v_A?

Let two reference frames S and S' coincide at time $t = 0$ and let S' rotate about the z axis so that at time t $\angle xOx' = \phi(t)$. Let the cartesian basis vectors of the respective systems by denoted by $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and $\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}'$, noting that $\boldsymbol{k} = \boldsymbol{k}'$.

3. Find the matrix A such that

$$\begin{bmatrix} \boldsymbol{i}' \\ \boldsymbol{j}' \end{bmatrix} = A \begin{bmatrix} \boldsymbol{i} \\ \boldsymbol{j} \end{bmatrix}.$$

4. Let $\boldsymbol{a} = (a_x, a_y, a_z)$ be an arbitrary vector. Find $\boldsymbol{a}$ in terms of the dashed coordinates.

5. We write $\frac{d\boldsymbol{a}}{dt}$ and $\frac{D\boldsymbol{a}}{dt}$ to denote time derivatives with respect to the stationary and the rotating frames respectively. Verify that

$$\frac{d\boldsymbol{a}}{dt} = \frac{D\boldsymbol{a}}{Dt} + \dot{\phi}(a'_x \boldsymbol{j}' - a'_y \boldsymbol{i}').$$

6. Writing $\boldsymbol{\Omega} = \dot{\phi}\boldsymbol{k}$, (the *angular velocity* vector), show that

$$\frac{d\boldsymbol{a}}{dt} = \frac{D\boldsymbol{a}}{dt} + \boldsymbol{\Omega} \times \boldsymbol{a}.$$

7(a) Take $\boldsymbol{a} = \boldsymbol{r}$, the position vector of a particle, in order to prove that

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{D^2\boldsymbol{r}}{Dt^2} + \frac{D\boldsymbol{\Omega}}{Dt} \times \boldsymbol{r} + 2\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}).$$

(b) Hence show that if a particle P is subject to a force $\boldsymbol{F}$ then in the rotating frame S' we have the equation of motion:

$$m\frac{D^2\boldsymbol{r}}{Dt^2} = \boldsymbol{F} - m\frac{D\boldsymbol{\Omega}}{Dt} \times \boldsymbol{r} - 2m\boldsymbol{\Omega} \times \frac{D\boldsymbol{r}}{dt} - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}).$$

8. Consider now the equation of part 7(b) as it applies to a particle of mass m on the Earth's surface under the action of an external force $\boldsymbol{F}$.

(a) Show that the equation simplifies to

$$\boldsymbol{F} = -m\boldsymbol{g}_e, \text{ where } \boldsymbol{g}_e = \boldsymbol{g} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega} \times \boldsymbol{r}).$$

(b) What is the direction of the vector $\boldsymbol{\Omega}\times(\boldsymbol{\Omega} \times \boldsymbol{r})$?

9. Show using the Cosine rule that for a point P of latitude λ on the Earth's surface, which has radius R, that

$$g_e^2 = g^2 + \Omega^4 R^2 \cos^2\lambda - 2g\Omega^2 R \cos^2\lambda.$$

10. Taking the radius of our planet to be $6,371$km, find the minimum effective acceleration at the Earth's surface as a percentage of that at the poles.

Problem Set 10: Central forces

1(a) Show that for any vector $\boldsymbol{a} = \boldsymbol{a}(t)$ that

$$\boldsymbol{a}\bullet\frac{d\boldsymbol{a}}{dt} = |\boldsymbol{a}|\frac{d|\boldsymbol{a}|}{dt}.$$

(b) Hence show that

$$\int m\ddot{\boldsymbol{r}} \bullet \dot{\boldsymbol{r}}\, dt = \frac{1}{2}m|\boldsymbol{v}|^2 + c.$$

2(a) By taking the scalar product of the equation of motion with $\dot{\boldsymbol{r}}$ for a force of the form $\boldsymbol{F}(\boldsymbol{r}) = F(r)\dot{\boldsymbol{r}}$ show that the energy equation takes the form:

$$\frac{1}{2}m|\boldsymbol{v}|^2 - \int F(r)\, dr = E.$$

(b) Find the form for the potential energy in the case where $F(\boldsymbol{r}) = \frac{1}{r^2}$.

3. The cartesian coordinates (x, y, z) of a moving particle of mass m are given by $x = at$, $y = bt^2$, and $z = ct^3$, where a, b, c are constants. Find expressions for

(a) the force acting on the particle;
(b) the moment of this forces about the origin O;
(c) the angular momentum of the particle about O.
(d) Verify that the moment of the force is equal to the rate of change of the angular momentum of the particle.

Consider a *central force*, acting on a point P, which is one of the form

$$\boldsymbol{F} = F(r)\hat{\boldsymbol{r}} = \frac{F(r)}{r}\boldsymbol{r}$$

where $\boldsymbol{r}$ denotes the position vector of P.

4(a) Show that the moment vector $\boldsymbol{M}$ about the origin is $\boldsymbol{0}$.
(b) Show that the angular momentum $\boldsymbol{L}$ of the particle is constant.

5(a) Prove that the path of P lies in a plane perpendicular to $\boldsymbol{L}$.
(b) Prove that the radial vector $\boldsymbol{r}$ sweeps out area at a constant rate.

6(a) Derive the identity:

$$\frac{d}{dt}(\dot{\boldsymbol{r}} \times \boldsymbol{L}) = \frac{F(r)}{r}\big((\boldsymbol{r} \bullet \dot{\boldsymbol{r}})\boldsymbol{r} - (\boldsymbol{r} \bullet \boldsymbol{r})\dot{\boldsymbol{r}}\big).$$

(b) And hence obtain:

$$\frac{d}{dt}(\dot{\boldsymbol{r}} \times \boldsymbol{L}) = F(r)\dot{r}\boldsymbol{r} - F(r)r\dot{\boldsymbol{r}}.$$

7. Show that if the right hand side of the equation of 6(b) may be written as a derivative of the form

$$\frac{d}{d\alpha}(\alpha(r)\boldsymbol{r})$$

then it is necessarily the case that $F(r) = -\frac{\lambda}{r^2}$, for some constant λ.

8(a) Use Question 7 to show that if $F(r)$ is an inverse square law then the vector

$$\boldsymbol{K} = \dot{\boldsymbol{r}} \times \boldsymbol{L} - \lambda\frac{\boldsymbol{r}}{r}$$

is constant.

(b) Show that, if non-zero, the vectors $\boldsymbol{L}$,$\boldsymbol{K}, \boldsymbol{L} \times \boldsymbol{K}$ form an orthogonal basis at each point P of the path of the particle.

9. If a particle of mass m moves under a central force $F(r) = -\mu r$ $(\mu > 0)$ show that

(a) $\boldsymbol{r} = \boldsymbol{a}\sin\omega t + \boldsymbol{b}\cos\omega t$, where $\omega^2 = \frac{\mu}{m}$ and $\boldsymbol{a}$ and $\boldsymbol{b}$ are arbitrary constant vectors.

(b) Find the angular momentum $\boldsymbol{L}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$, μ and m;

(c) Similarly determine the total energy E.

10. A particle of mass m moves under the action of an inverse square force $F(r) = -\frac{\lambda\boldsymbol{r}}{r^3} = -\frac{\lambda\hat{\boldsymbol{r}}}{r^2}$. Show directly that the time derivatives of both the angular momentum vector $\boldsymbol{L} = m\boldsymbol{r} \times \dot{\boldsymbol{r}}$ and $\boldsymbol{K} = \dot{\boldsymbol{r}} \times \boldsymbol{L} - \lambda\frac{\boldsymbol{r}}{r}$ are both $\boldsymbol{0}$.

Hints for Problems

Problem Set 1

1. Solve $\sum_k m_k(x_k - \overline{x}) = 0$.

Problem Set 2

Apply the definitions and results stated in the introduction to Set 1.
3. Switch to polar coordinates.
4. $I_x = \int r^2 \, dm$.
6(a) $I = \int \int \int_R \rho x^2 \, dV$.
7(a) $dV = lrdrd\theta$.
(b) Use (a) and Perpendicular axis theorem.
(c) Use (b), integrate contribution of thin slices parallel to base, and employ the Parallel axis theorem.
10. Find dm for thin circular discs (remember the result of 7(a)) in terms of m, R, h and r and use similar triangles to express r in terms of z.

Problem Set 3

9 & 10. Do make use of the symmetries of these shapes to simplify integrals.

Problem Set 4

2(a) You need to introduce a symbol R for the radius of the pulley in order to work with the relevant equations, but it cancels out in the end.
7. Equate the torque around the suspension point to 0, noting that the weight of the beam acts through its centre of mass.

Problem Set 5

6. Test each case and see which one give an answer consistent with the given data.

7(b) If m_2 moves a distance x downwards, how far does m_1 move upwards?

8(a) This time $T_1 = 2T_2$ and the acceleration of the second pulley is the average of that of m_2 and m_3; but you need to justify both these facts.

10(b) The trickiest equation is $T_2 - T_1 - m_2\mu g = m_2 a$, then use (a).

Problem Set 6

5(b) Modify the virtual work equation, adjoining a term for the virtual work done by the frictional force R.

10. Find R_B using virtual work. Then find R_A may be found easily from Newton's law.

6. $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\bullet\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\bullet\boldsymbol{b})\boldsymbol{c}$.

Problem Set 7

7. We require that $T \geq 0$ so write down an equation for the tension in the string using Newton's law and use conservation of energy to replace v^2 with the given parameters of the equation. In (b) $T < 0$ is now permitted so we need only ensure that $v > 0$, which is a weaker condition.

9. Use energy conservation to find out the value of $a\ddot{\theta}$. Then use Newton's laws to write down equations for the frictional force F and the vertical reaction R. Finally put $\mu = \frac{F}{R}$ to find the value of μ at the slipping point.

10. There is a rebound impulse P perpendicular to the door and an impulse J at the hinge A also perpendicular to the door. The moment imparted by the impulse equals the change in angular momentum of the door. Equate the angular momentum before and after collision and put $J = 0$.

Problem Set 8

7. Think *spherical coordinates.*

Problem Set 10

6(a) Show that $\dot{\boldsymbol{r}} \bullet \dot{L} = 0$, (b) Use that $\frac{d}{dt}(\boldsymbol{r} \bullet \boldsymbol{r}) = 2r$.

Answers to the Problems

Problem Set 1

1. $\frac{\sum_k m_k x_k}{\sum_k m_k}$. 2. $(\overline{x}, \overline{y}) = (\frac{1}{2}a, \frac{13}{18}a)$. 3(b) $\left(\frac{7}{6}, \frac{23}{6}\right)$. 4 & 5. $(\frac{2a}{3}, \frac{b}{3})$. 6. $(0, \frac{4a}{3\pi})$. 7. $(0, \frac{8}{5})$. 8. $(0, \frac{8}{7})$. 9. $M = 14$, $M_x = 11$, $M_y = 10$, $(\overline{x}, \overline{y}) = (\frac{5}{7}, \frac{11}{14})$. 10. $I_x = 12$, $I_y = \frac{39}{5}$, $I_0 = \frac{99}{5}$; $R_x = \sqrt{\frac{6}{7}}$, $R_y = \sqrt{\frac{39}{5}}$, $R_0 = \sqrt{\frac{99}{70}}$.

Problem Set 2

1. $(\frac{1}{2}, \frac{2}{5})$. 2. $\frac{M}{12}(b^2 + c^2)$. 3. $(0, 0, \frac{4}{3})$. 6(a) $\frac{ml^2}{3}$, (b) $\frac{ml^2}{12}$ 7(a) $\frac{mR^2}{2}$ (b) $\frac{mR^2}{4}$ (c) $\frac{m}{12}(3R^2 + h^2)$.8(a) $\frac{mb^2}{12}$, (b) $\frac{mb^2}{3}$. 9(a) $\frac{m}{12}(a^2 + b^2)$, (b) $\frac{m}{3}(a^2 + b^2)$, (c) $\frac{m}{12}(4a^2 + b^2)$. 10. $\frac{3}{10}mR^2$.

Problem Set 3

1(a) $\frac{2}{5}mR^2$, (b) $\frac{2m}{5} \cdot \frac{R_1^5 - R_2^5}{R_1^3 - R_2^3}$. 2. $\frac{2}{3}mR^2$. 4. $\sqrt{\frac{10gh}{7}}$ 5. $\sqrt{\frac{6gh}{5}}$ 6. $\sqrt{\frac{4gh}{3}}$; sphere beats cylinder beats hollow sphere. 7. $\sqrt{gh}$. 9. $\frac{(3\pi+4)a}{18\pi}$. 10. $\frac{128}{105\pi}$.

Problem Set 4

1. $1884 \cdot 96J$, $157 \cdot 08$ watts. 2(a) $\frac{1}{2}Ma$, (b) $\frac{2mg}{2m+M}$, (c) $2\sqrt{\frac{mgh}{2m+M}}$. 4(a) $T_1 = m_1 a$ and $m_2 g - T_2 = m_2 a$, (b) $\frac{2m_2 g}{2m_1+2m_2+M}$. 5(a) $\frac{2m_2 - 2m_1(\sin\theta + \mu\cos\theta)}{M+2m_1+2m_2}g$, (b) $m_2 = m_1(\sin\theta + \mu\cos\theta)$. 6(a) $g\sin\theta - g\mu\cos\theta$, (b) $\frac{a}{2g\cos\theta}$, (c) $\frac{2}{3}g\sin\theta$, (d) $\arctan(3\mu)$. 7. $2m\left(1 + \frac{a}{L-2a}\right)$. 8(a) $\frac{2g}{2+\left(\frac{R}{r}\right)^2}$, (b) $\frac{2}{3}g$. 9. $R_B = \frac{a}{L} \cdot mg$, $R_A = (1 - \frac{a}{L})mg$. 10.

Problem Set 5

1. $\mu_s = 0\cdot 4$, $\mu_k = 0\cdot 2$. 2(a) $-7\cdot 84\text{m/sec}^2$, (b) $25\cdot 5\text{m}$, (c) $2\cdot 55\text{s}$. 3(a) $\frac{F(\cos\theta+\mu\sin\theta)-mg\mu}{m}$, (b) $\frac{mg\mu_s}{\cos\theta+\sin\theta\mu_s}$. 4(a) $a = \frac{g(m-\mu M)}{M+m}$; $M > \mu m$, (b) $\frac{1}{5}$. 5(a) $g\frac{m-M(\sin\theta+\mu\cos\theta)}{M+m}$, (b) $g\frac{M(\sin\theta-\mu\cos\theta)-m}{M+m}$. 6. $0\cdot 9$ m/sec^2. 7(a) $2T - m_1 g = m_1 a$, $T - m_2 g = m_2 a_1$, (b), $a_2 = 2a_1$ (c) $a_1 = \frac{g(2m_2-m_1)}{4m_2+m_1}$, $a_2 = \frac{2g(2m_2-m_1)}{4m_2+m_1}$, $T = \frac{3m_1 m_2 g}{4m_2+m_1}$. 8(b) $\begin{bmatrix} m_1 & 0 & 2 \\ 0 & m_2 & 1 \\ -2m_3 & -m_3 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ T_2 \end{bmatrix} = g\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}$. 9(a) $m_1m_2 + m_1m_3 + 4m_2m_3$, (b) see worked solutions, (c) $a_1 = -\frac{g}{3}$, $a_2 = a_3 = \frac{g}{3}$, $T_1 = \frac{4gm}{3}$, $T_2 = \frac{2gm}{3}$. 10(a) $\frac{F}{m_1+m_2+m_3} - \mu g$, (b) $T_1 = \frac{m_1 F}{m_1+m_2+m_3}$; $T_2 = \frac{(m_1+m_2)F}{m_1+m_2+m_3}$.

Problem Set 6

1. & 2. $R_B = \frac{a}{L}\cdot mg$, $R_A = (1-\frac{a}{L})mg$. 3(a) $y = L\cos\theta$, $\frac{dy}{d\theta} = -L\sin\theta$, $x = 2L\sin\theta$, $\frac{dx}{d\theta} = 2L\cos\theta$; (b) 5(a) $\frac{F\mu}{2}$ (b) $.B_X = \frac{F}{2}\big(\tan\theta - \mu\big)$, (c) $1-\mu\cot\theta$. 6(a) $\tan\theta = \frac{d}{y}, y = d\cot\theta$. $\frac{dy}{d\theta} = -\frac{d}{\sin^2\theta}$, (b)$\delta U = -P\delta y - M\delta\theta$, (c) $\frac{Pd}{\sin^2\theta}$. 7(a) one; generalized coordinate can be either x or θ, each of which determine the position of the system. (b) $y_P = \frac{l}{2}\cos\frac{\theta}{2}$, $\delta y_P = -\frac{l}{4}\sin\frac{\theta}{2}$(b) $x_P = l\sin\frac{\theta}{2} + \frac{l}{2}\sin\frac{\theta}{2} = \frac{3}{2}l\sin\frac{\theta}{2}$,$\delta x = \frac{3}{4}l\cos\frac{\theta}{2}\delta\theta$. (c)$\theta = 2\arctan\frac{3P}{2mg}$. 8. $y_C = 2l\sin\theta$, $y_b = \frac{1}{2}y_c$. (b) $F = ks = 2kl\sin\theta - kh$. (c) $\arcsin\Big(\frac{P+2kh}{4kl}\Big)$. 9. $2mg\cot\theta\sqrt{1+(\frac{b}{L})^2 - \frac{2b}{L}\cos\theta}$. 10. $R_B = 6\sqrt{3} - \frac{24\sqrt{2}}{5} = 3\cdot 6041$ (4d.p),$R_A = 4\sqrt{3} + \frac{4\sqrt{2}}{5} = 8\cdot 0596$ (4 d.p.).

Problem Set 7

5(a) $\theta(t) = C_1\cos\sqrt{\frac{g}{l}}t + C_2\sin\sqrt{\frac{g}{l}}t$, (b) $\theta(t) = \theta_0\cos\sqrt{\frac{g}{l}}t,\ t\geq 0$, (c) $\frac{2\pi}{\sqrt{g}}\sqrt{l}$. 7(a)$\sqrt{\frac{5g}{l}}$, (b) $\sqrt{\frac{4g}{l}}$. 8(b) $X = mg\sin\theta + ma\dot\theta^2 = \frac{5}{2}mg\sin\theta$, $Y = mg\cos\theta - ma\ddot\theta = \frac{1}{4}mg\cos\theta$. 9. $\mu = \frac{6(3\sqrt{3}-4)}{31-12\sqrt{3}} = 0\cdot 3513$ (4 d.p.). 10. $\frac{4}{3}a$.

Problem Set 8

1(a) 1 degree of freedom; (b) $L = \frac{m}{2}(\dot x^2 + \dot y^2)$ (c) $L(r,\dot r,t) = \frac{m}{2}(\dot r^2 + r^2\omega^2)$. 2. $r(t) = C_1 e^{\omega t} + C_2 e^{-\omega t}$. 3(a) $L = ma^2\dot\theta^2(1-\cos\theta) - mga(1+\cos\theta)$, (b)$\ddot\theta(1-\cos\theta) + \frac{1}{2}\dot\theta^2\sin\theta - \frac{g}{2a}\sin\theta = 0$. 4. $\ddot\theta + \dot\theta^2\cot\frac{\theta}{2} - \frac{g}{2a}\cot\frac{\theta}{2} = 0$. 5. $\cos(\frac{\theta}{2}) =$

$C_1 \cos\sqrt{\frac{g}{4a}}t + C_2 \sin\sqrt{\frac{g}{4a}}t$. 7(a) 1, (b)$x = R\sin\theta\cos\omega t,\ y = R\sin\theta\sin\omega t,\ z = R\cos\theta$, $\dot{x} = R\dot{\theta}\cos\theta\cos\omega t - R\omega\sin\theta\sin\omega t,\ \dot{y} = R\dot{\theta}\cos\theta\sin\omega t + R\omega\sin\theta\cos\omega t,\ \dot{z} = -R\dot{\theta}\sin\theta$. (c) $L = \frac{1}{2}mR^2(\dot{\theta}^2 + \omega^2\sin^2\theta) - mgR\cos\theta$. 8. $\ddot{\theta} = \frac{1}{2}\omega^2\sin 2\theta + \frac{g}{R}\sin\theta$. 9. 2 degrees of freedom; $x_m = x_M + q\cos\alpha,\ y_m = h - q\sin\alpha, \dot{x}_m = \dot{x}_m + \dot{q}\cos\alpha,\ \dot{y}_m = -\dot{q}\sin\alpha$. (b) $L = \frac{1}{2}m(\dot{x}_M^2 + \dot{q}^2 + +2\dot{x}_M\dot{q}\cos\alpha) + \frac{1}{2}M\dot{x}_M^2 - mg(h - q\sin\alpha)$. 10(a) $\ddot{x}_M = -\frac{mg\sin\alpha}{M+m\sin^2\alpha}$; $\ddot{q} = g\sin\alpha + \frac{mg\sin\alpha}{M+m\sin^2\alpha}$, $\ddot{x}_m = \ddot{q} + x_M = g\sin\alpha$.

Problem Set 9

1(a) $mr\omega^2$, (b) $2m\omega v_r$. 2(a) $v_A + \omega r$, (b) $\frac{v_A^2}{r} + 2\omega v_A + \omega^2 r$. 3. $\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$. 4. $= (a'_x\cos\phi - a'_y\sin\phi)\boldsymbol{i} + (a'_x\sin\phi + a'_y\cos\phi)\boldsymbol{j} + a'_z\boldsymbol{k}$. 5.

Problem Set 10

3(a) $m(0,\ 2b,\ 6ct)$. (b) $m(4bct^3,\ -6act^2,\ 2abt)$ (c) $m(bct^4,\ -2act^3,\ abt^2)$. 9(b) $\boldsymbol{L} = m\omega(\boldsymbol{b} \times \boldsymbol{a})$, (c) $E = \frac{1}{2}m\omega^2(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2) = \frac{\mu}{2}(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2)$.