Mathematical fancy dress

Everyone enjoys wearing a new outfit from time to time and so here we dress up a well-known friend from calculus in algebraic attire. Whether or not you approve the new look, there is at least an element of novelty in this month's problems.

Problem 1 Show that the open interval $(-1, 1)$ is a group G under the binary operation $\circ$ defined by the rule,

$$x \circ y = \frac{x + y}{1 + xy}. \tag{1}$$

This is a nice exercise but there is a bit to check for we cannot take associativity for granted and even before that we need to verify that (1) truly represents a binary operation in that the output lies in $(-1, 1)$.

This may all look rather mysterious, which leads to a second question.

Problem 2 Our group $(G, \circ)$ is a copy of which very well-known abelian group?

Solution to Mathematical fancy dress

Problem 1 First we check that $(-1, 1)$ is closed under the binary operation

$$x \circ y = \frac{x+y}{1+xy}.$$

Let $x, y \in (-1, 1)$, and so $1 + xy > 0$. Then

$$x \circ y < 1 \Leftrightarrow x + y < 1 + xy \Leftrightarrow xy - x - y + 1 > 0$$

$$\Leftrightarrow (x-1)(y-1) > 0,$$

which is true as both terms in the product are negative. Similarly

$$x \circ y > -1 \Leftrightarrow x + y > -1 - xy \Leftrightarrow xy + x + y + 1 > 0$$

$$\Leftrightarrow (x+1)(y+1) > 0,$$

which is true as both terms are positive. It follows that $x \circ y \in (-1, 1)$ and so $\circ$ is a (commutative) binary operation. Next we check associativity of $\circ$. For $x, y, z \in (-1, 1)$ we get:

$$(x \circ y) \circ z = \frac{x+y}{1+xy} \circ z = \frac{\frac{x+y}{1+xy} + z}{1 + \frac{(x+y)z}{1+xy}} = \frac{x+y+z+xyz}{1+xy+yz+zx}. \ (*)$$

On the other hand, by commutativity we have $x \circ (y \circ z) = (y \circ z) \circ x = (z \circ y) \circ x$ and since the RHS of (*) is symmetric in the three variables, it follows that $(x \circ y) \circ z = x \circ (y \circ z)$ and hence $(G, \circ)$ is a commutative semigroup. What is more, $x \circ 0 = \frac{x+0}{1+(x)(0)} = x$, so that 0 is the identity element, whence $(G, \circ)$ is a monoid. Finally, $-x$ is the inverse of x with respect to the operation $\circ$ as $x \circ (-x) = \frac{x+(-x)}{1+x(-x)} = \frac{0}{1-x^2} = 0$ (as $x^2 \neq 1$). Therefore $(G, \circ)$ is indeed an abelian group.

Problem 2 The group $(G, \circ)$ is isomorphic to $(\mathbb{R}, +)$. A required isomorphism is $\tanh : \mathbb{R} \to G$. The hyperbolic tangent function maps onto $(-1, 1)$ as $\lim_{x\to\pm\infty} \frac{\sinh(x)}{\cosh(x)} = \pm 1$. Also $(\tanh(x))' = \operatorname{sech}^2(x) > 0$, and so $\tanh(x)$ is one-to-one. Finally $\tanh(x)$ is a group homomorphism as through applying a standard identity we see that

$$\tanh(x+y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)} = \tanh(x) \circ \tanh(y).$$