Mathematics 203 Vector Calculus

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Vector Calculus is the foundation of mathematical physics. It follows on from the calculus of several variables but importantly introduces what is sometimes referred to as ‘div, grad, curl, and all that’, which explore the relationships between *scalar* (real-valued) *fields*, such as temperature of points in space, and *vector fields,* which are mappings from $\mathbb{R}^3$ to $\mathbb{R}^3$, such as wind speed and direction at points in the atmosphere.

The first problem set revises and enhances the ideas of *scalar* and *cross products* of vectors. Set 2 is about the *gradient operator*, which associates a vector field to a scalar field. The following sets introduce the *divergence operator*, which associates a scalar field with a given vector field, and the *curl*, which is another vector field associated with a vector field that represents its tendency to cause rotation. These operators are related by a number of important identities, which are set as exercises. All such identities rely on the smoothness of the vector field - in particular they require that $f_{xy} = f_{yx}$, which is to say that the order in which partial differentiation is carried out does not affect the outcome. The justificiation for these analytical assumptions will be dealt with in our module on real analysis. In particular, results such as the one just mentioned rely for their proof on the *mean value theorem* in various forms.

We next return to *line intergrals*, which were introduced in MA201 and in particular we consider so called *conservative vector fields*, for which the value of a line integral from one point to another is independent of path taken between them.

The later problem sets introduce *Green's theorem,* the *Divergence theorem*, and *Stokes's theorem*, each of which equate the value of an integral to a related integral on a set of lower dimension. As such, each is a generalization of the Fundamental theorem of calculus and for that reason the proof of each has an argument that eventually falls back on to the Fundamental theorem.

The final set introduces the *Frenet-Serret* equations which govern the behaviour of smooth curves in 3-space.

Problem Set 1 Revision of dot and cross products

Verify the following properties of *dot* and *cross products* (also known as *scalar* and *vector* products respectively); $\mathbf{a}, \mathbf{b}, \mathbf{c}$ etc. denote arbitrary vectors in the plane or in three dimensions as the case may be.

1. Show that

$$\mathbf{a} \bullet \mathbf{b} = ||\mathbf{a}||\,||\mathbf{b}||\,\cos\theta$$

where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ in 3-space ($\mathbb{R}^3$).

2. Hence deduce the *Cauchy-Schwarz Inequality*

$$(x_1y_1 + x_2y_2)^2 \leq (x_1^2 + y_1^2)(x_2^2 + y_2^2).$$

3. Use dot products to prove the *Triangle Inequality*

$$||\mathbf{a} + \mathbf{b}|| \leq ||\mathbf{a}|| + ||\mathbf{b}||.$$

4. Verify that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

5. Show that $\mathbf{a} \bullet (\mathbf{a} \times \mathbf{b}) = 0$.

6. Verify that

$$||\mathbf{a} \times \mathbf{b}||^2 = ||\mathbf{a}||^2||\mathbf{b}||^2 - (\mathbf{a} \bullet \mathbf{b})^2.$$

7. Use Question 6 to show that

$$||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}||\,||\mathbf{b}||\,\sin\theta$$

where θ is the angle between the two vectors and that $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are parallel.

8. Show that

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}).$$

9. Show that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{c}$$

10. Deduce from Question 9 that the cross product is in general *not* associative and find necessary and sufficient conditions on $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ under which $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

Problem Set 2 The Gradient vector field of a scalar field

1. If $f : \mathbb{R}^3 \to \mathbb{R}$ is a non-constant function (a *scalar field*) show that the *gradient vector* $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$ is orthogonal to the *level surface* $f(x, y, z) = c$ (c a constant).

2. For f as in Question 1, show that $f(x, y, z)$ increases fastest in the direction of $\nabla f(x, y, z)$ and decreases fastest in the opposite direction.

3. The temperature $T(x, y, z)$ of a certain object measured with the origin at its centre of mass is given by

$$T(x, y, z) = e^{-x} + e^{-2y} + e^{4z}.$$

In which direction from the point $(1, 1, 1)$ will the solid be cooling the fastest?

4. Find the *directional derivative* of the scalar field

$$f(x, y, z) = x^2yz + 4xz^2$$

at the point $(1, -2, -1)$ in the direction $(2, -1, -2)$.

5. Find a unit normal $\mathbf{n}$ to the surface $x^2 + y^2 - z = 0$ at the point $(1, 1, 2)$. Is the normal unique?

6. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the common point $(2, -1, 2)$.

7. Find $\nabla f(2, 3)$ where $f(x, y) = \frac{x}{x^2+y^2}$.

8. Find the equation of the tangent plane to the surface $z = \frac{x}{x^2+y^2}$ at the point corresponding to $(2, 3)$.

9. For arbitary scalars λ and μ, show that

$$\nabla(\lambda f + \mu g) = \lambda \nabla f + \mu \nabla g.$$

10. Prove the product rule for the gradient operator:

$$\nabla(fg) = (\nabla f)g + f(\nabla g).$$

Problem Set 3 Divergence and Curl of a vector field

We write ∇ to denote the *differential operator* $\nabla=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})$. The *divergence* div$\mathbf{F}$ of a vector field $\mathbf{F}=(f_1,f_2,f_3)$ is then the scalar field $\nabla\bullet\mathbf{F}$. On the other hand the *curl*($\mathbf{F}$) is another vector field *curl* $\mathbf{F}=\nabla\times\mathbf{F}$. The *Laplacian* of a scalar field $f(x,y,z)$ is $\nabla^2 f=\nabla\bullet\nabla f$. We shall also sometimes use the notation f_x, f_y etc. as an alternative to $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$ etc.

1. Show that

$$\mathrm{div}(\mathbf{F})=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}.$$

2. For $\mathbf{F}(x_1,x_2,x_3)=(f_1,f_2,f_3)$ write curl($\mathbf{F}$) in both determinant form and explicitly as a vector field in terms of its *component functions* f_1,f_2 and f_3. Show that curl$(F)=\mathbf{0}$ if and only if

$$\frac{\partial f_i}{\partial x_j}=\frac{\partial f_j}{\partial x_i}\,\forall 1\le i,j\le 3.$$

3. Find the divergence of $\mathbf{F}(x,y,z)=(xyz,xz,z)$.

4. And now find the curl of the vector field of Question 3.

5. Find the divergence of

$$\mathbf{F}(x,y,z)=\ln x\mathbf{i}+e^{xyz}\mathbf{j}+\tan^{-1}\left(\frac{z}{x}\right)\mathbf{k}.$$

6. And now find the curl of the vector field of Question 5.

7. Write the Laplacian of $f(x,y,z)$ explicitly as a scalar field in terms of f and its second order partial derivatives.

8. Show that the function

$$\phi(x,y,z)=\sin kx\sin ly e^{\sqrt{k^2+l^2}z}$$

is *harmonic*, meaning that it satisfies *Laplace's equation* $\nabla^2\phi=0$.

9. Show that the curl of the gradient of a scalar field $f(x,y,z)$ is $\mathbf{0}$, which is to say

$$\nabla\times\nabla f=\mathbf{0}.$$

10. Show that the divergence of the curl is 0, which is to say that for a vector field $\mathbf{F}$

$$\nabla\bullet(\nabla\times\mathbf{F})=0.$$

Problem Set 4 Line Integrals, Conservative Fields

1. For vector fields $\mathbf{f}$ and $\mathbf{g}$ show that

$$\nabla \bullet (\mathbf{f} \times \mathbf{g}) = g \bullet (\nabla \times \mathbf{f}) - \mathbf{f} \bullet (\nabla \times \mathbf{g}).$$

2. Let f and g denote scalar fields. Use the identity of Question 1 to show that $\nabla f \times \nabla g$ is *solenoidal*, which is to say that

$$\nabla \bullet (\nabla f \times \nabla g) = 0.$$

3. The curl of the curl is another vector field. Show that it is governed by the identity:

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \bullet \mathbf{v}) - \nabla^2 \mathbf{v};$$

the first ∇ on the RHS indicates the taking of the gradient of the scalar field in the bracket. The *Laplacian* ∇^2 is understood to operate componentwise on each of the component functions of the vector field $\mathbf{v}$. It is enough here to verify equality of first components, as the others follow in the same way.

4. Find the work done by the vector field $\mathbf{F}(x, y, z) = (y - x^2)\mathbf{i} + (9z - y)\mathbf{j} + (x - z^2)\mathbf{k}$ along the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $0 \leq t \leq 1$.

5. A vector field $\mathbf{F}(x, y, z)$ has a *potential function* ϕ if $\mathbf{F}(x, y) = \nabla\phi$ for some scalar field ϕ. Express the vector field

$$\mathbf{F}(x, y) = (x^2 + y^2, 2xy + 1)$$

in the form $\nabla\phi$ for a suitable potential ϕ.

6. Show that $\mathbf{u}(x, y, z) = (x+2y+4z, 2x-3y-z, 4x-y+2z)$ has a potential function $\phi(x, y, z)$ by solving the defining equation.

7. Find the curl of $\mathbf{u}(x, y, z) = (xyz, x, z)$ and deduce that $\mathbf{u}$ is not the gradient of any potential function.

8. Suppose that $\mathbf{f}(x, y, z) = \nabla\phi$. Show by using the Chain rule and the definition of line integral that

$$\int_C \mathbf{f}\, ds = \phi(\mathbf{b}) - \phi(\mathbf{a})$$

for any parametrizable curve C from points with position vectors $\mathbf{a}$ and $\mathbf{b}$. Conclude that if C is a simple closed curve then

$$\oint_C \mathbf{f}\, ds = 0,$$

where $\oint_C$ denotes the integral around the closed curve. A vector field that has line integrals independent of path in this fashion in known as a *conservative field*.

9. Use the calculation of Question 5 and the result of Question 8 to find $\int_C (x^2 + y^2, 2xy + 1)\, ds$ where C is any smooth curve from the origin to the point $(1, 2)$.

10. Calculate the line integral of Question 9 directly for the case where C is the straight line segment L from the origin to $(1, 2)$.

Problem Set 5 Circulation and Green's theorem

1. Evaluate the integral

$$\int_C (x^2 + y^2, 2xy + e^{-y}) \bullet d\mathbf{r}$$

where C is a smooth curve from $(1, -1)$ to $(0, 8)$ by first finding a potential ϕ for the vector field $\mathbf{F}$ to be integrated.

2. Find the potential function $\phi(x, y, z)$ for the vector field

$$\mathbf{F}(x, y, z) = (z \cos x + \ln z, y^2, \sin x + \frac{x}{z}),$$

which satisfies the initial condition that $\phi(\frac{\pi}{2}, 1, 1) = 0$.

3. By using a suitable parametrization, calculate

$$\oint_C (2y\, dx - 3x\, dy)$$

traversed anti-clockwise around the circle C with equation $x^2 + y^2 = 1$.

4. For a space curve

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

if $\mathbf{v} = \omega \times \mathbf{r}$, prove that $\omega = \frac{1}{2}\text{curl}\mathbf{v}$, where $\omega = (\omega_1, \omega_2, \omega_3)$ is a constant vector (called the *angular velocity*).

5. By considering the line integral of $\mathbf{F}(x, y, z) = (y, x^2 - x, 0)$ around the unit square in the plane connecting the points $(0, 0)$,$(1, 0)$,(1,1) and $(0, 1)$ in that order, show that $\mathbf{F}$ is not a conservative field.

6. Let $\mathbf{F}(x, y, z) = -\nabla\phi$ be a conservative force field. Suppose a particle of mass m moves in this field. If A and B are any two points in space, we have

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \phi(A) - \phi(B)$$

where C is any path from A to B. (Note the change in sign as we have written the potential in the form $-\nabla\phi$, a convention in mechanics.) By use of Newton's Law, $\mathbf{F} = m\mathbf{a}$ re-write this integral to conclude the L*aw of Conservation of energy* in the form:

$$\phi(A) + \frac{1}{2}mv_A^2 = \phi(B) + \frac{1}{2}mv_B^2$$

where v_A and v_B are the respective velocities of the mass at points A and B.

7. An alternative type of line integral that results in a vector solution is

$$\int_C \mathbf{F} \times d\mathbf{r} = \int_{t=a}^{b} \mathbf{F}(\mathbf{r}(t)) \times \mathbf{r}'(t)\, dt.$$

Calculate this integral where C is the curve $\mathbf{r}(t) = (t^2, 2t, t^3)$ and $\mathbf{F}(x, y, z) = (xy, -z, x^2)$ for $0 \leq t \leq 1$.

Green's Theorem Let $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a vector field. Then for a closed curve C enclosing a region R, with C traversed with the R on the left,

$$\oint_C \mathbf{F} \bullet d\mathbf{r} = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

8. Use Green's Theorem to evaluate the integral of Question 3 (taking the formula for the area of a circle for granted.)

9. Evaluate

$$\oint_C \mathbf{F} \bullet d\mathbf{r} = \oint_C y^3 dx - x^2 dy$$

where C is a positively oriented circle of radius 2 centred at the origin.

10. Find a simple closed curve that maximizes the value of

$$\int_C \frac{y^3}{3}\, dx + \left(x - \frac{x^3}{3}\right) dy$$

and find that maximum value.

Comment The crux of the proof of Green's theorem is to show that

$$\int_C P(x, y)\, dx = -\int\int_R \frac{\partial P}{\partial y}\, dxdy, \quad \int_C Q(x, y)\, dx = \int\int_R \frac{\partial Q}{\partial x}\, dxdy.$$

In order to do this, the boundary curve C is expressed as the graph of two functions, which allows $x = t$ to be used in the parametrization. The difference that arises in the integral is then written, by the Fundamental theorem of calculus, as the evaluation of the integral of the stated partial derivatives. In this way Green's theorem and others such as the Divergence theorem and Stokes theorem, are all generalizations of the Fundamental theorem to the calculus of several variables.

Problem Set 6 Green's theorem examples

1. Use Green's theorem to find the value of $\oint_C \mathbf{F}\bullet d\mathbf{r}$ where $\mathbf{F}(x,y) = (x^2y, 2xy)$ and where C is the closed curve that bounds the region

$$R = \{(x,y) : 0 \leq x \leq 1,\, x^2 \leq y \leq x\}.$$

2. Evaluate the integral of Question 1 directly.
3. Use Greens' theorem to evaluate

$$\int_c x^2y\, dx - y^2x\, dy$$

where C is the circle $x^2 + y^2 = 4$.
4. Evaluate the integral of Question 3 directly.
5. Use Green's theorem to show that the area A of a region R in the plane bounded by a closed curve C is given by

$$A = \frac{1}{2}\oint_C xdy - ydx.$$

6. Use the formula of Question 5 to find the area of an ellipse with major axis of length $2a$ and minor axis of length $2b$.
7. Let C be a line segment from (a,b) to (a,c). Show that

$$\int_C xdy - ydx = ad - bc.$$

8. Use Question 7 and Green's theorem to find the area of a polygon with successive vertices $(x_1,y_1),(x_2,y_2),\cdots,(x_n,y_n)$. Use this result to find the area of the quadrilateral with vertices $(0,0),(3,4),(-2,2),(-1,0)$.

For Green's theorem to hold, the vector field being integrated needs to be differentiable throughout the region R, as is demonstrated in the following example.

9. Calculate $\oint_C \mathbf{F}\bullet d\mathbf{r}$ where

$$\mathbf{F}(x,y) = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$$

and C is the unit circle centred at the origin.
10. For the field $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ and the closed curve C of Question 9, calculate

$$\int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

Note that the answers to Questions 9 and 10 do not agree so Green's theorem does not hold here, and that $\mathbf{F}$ has a singularity at the origin.

Problem Set 7 Surface integrals

The *integral of a scalar field* $f(x, y, z)$ *over a surface* S parametrized by $\mathbf{r}(u, v)$ as u and v range over some region R of the uv-plane is given by:

$$\int\int_S f(x, y, z)\, d\sigma = \int\int_R f(x(u, v), y(u, v), z(u, v))\, \|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\|\, dudv. \quad (1)$$

The *integral of a vector field* $\mathbf{F}(x, y, z)$ *in the direction* $\mathbf{n}$ *normal to a surface* S represents the total *flux* of $\mathbf{F}$ through S and is calculated through parameterizing the surface:

$$\int\int_S \mathbf{F} \bullet \mathbf{n}\, d\sigma = \int\int_W \mathbf{F}(u, v) \bullet \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\, dudv \quad (2)$$

where W is the region in the uv-plane that gives the new limits. (The symbol Σ is often used instead of S so it is natural to write $d\sigma$ for the increment of surface area - since ds is the symbol for increment of arc length, their is a reluctance to use this symbol as then ds has two different meanings, although in context there is no ambiguity.)

The vectors $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ are tangent vectors to the surface at the point corresponding to (u, v) and so their vector product is normal to that surface at that point. The increment of surface area is a parallelogram with these vectors as sides, the area of which is the length of their cross-product so that the term $\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\|$ is the multiplier introduced into the integrand when we parametrize with the variables u and v. Applying this observation to the integral on the left in (2), we see that the term $\mathbf{F} \bullet \mathbf{n}$ takes the form

$$\mathbf{F} \bullet \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} / \|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\|$$

and so the length term in the denominator is cancelled by the same term in $d\sigma = \|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\|\, dudv$, the increment of surface area.

Formula (1) can also be recovered from (2): if we take $\mathbf{F}$ to be the vector field $\mathbf{F} = f(x, y, z)\mathbf{n}$, then the LHS of (2) becomes the RHS of (1), so that (1) can be thought of as the special case of integration of a vector field that is always acting orthogonally to the tangent plane of the surface of integration.

1. Calculate the surface integral of $f(x, y, z) = x^2$ over the surface of the cone $z = \sqrt{x^2 + y^2}$ for $0 \leq z \leq 1$.

2. Suppose that a surface S is given by the equation $z = g(x, y)$. Show that (1) can now be re-written in the form

$$\int\int_S f(x, y, z)\, d\sigma = \int\int_R f(x, y, g(x, y)) \sqrt{1 + (\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2}\, dxdy. \quad (3)$$

3. Show that the formula in Question 2 can also be written as

$$\int\int_S f(x,y,z)\,d\sigma = \int\int_R f(x,y,g(x,y))||\nabla h||\,dxdy \tag{4}$$

where $h(x,y,z) = z - g(x,y)$.

4. Answer Question 1 again this time using (3) instead of (1).

5. Explicitly introduce $h(x,y,z)$ and use the formula (4) to evaluate the surface integral for the scalar field

$$f(x,y,z) = x + 2y + 3z$$

over S, which is the upper surface of the part of the plane $x + y = 1$ that lies in the positive octant between $z = 0$ and $z = 1$ (the *positive octant* means the volume of 3-d space where $x, y, z \geq 0$).

6. Use a surface integral to show that the area of a right circular cone of radius R and height h is $\pi R\sqrt{h^2 + R^2}$.

7. For the function $g(u,v) = (u\cos v, u\sin v, v)$ $(0 \leq u \leq 1,\ 0 \leq v \leq 3\pi)$, show that

$$||\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}|| = \sqrt{1+u^2}.$$

By expressing it as an appropriate surface integral, find the area of the spiral ramp represented by the surface $g(u,v)$. You may use the fact that

$$\int \sqrt{1+x^2}\,dx = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln(x + \sqrt{1+x^2}).$$

8. Calculate the surface integral of the vector field $\mathbf{F}(x,y,z) = (x, z, -y)$ over the surface of the cylinder $S = \{(x,y,z) : x^2 + y^2 = 1,\ 0 \leq z \leq 1\}$.

9. Show that if S is defined by $z = g(x,y)$ then

$$\int\int_S \mathbf{F}(x,y,z) \bullet \mathbf{n}\,d\sigma = \int\int_R \mathbf{F} \bullet \nabla h\,dxdy$$

where $h(x,y,z) = z - g(x,y)$.

10. Evaluate the surface integral of the vector field

$$\mathbf{F}(x,y,z) = (18z, -12, 3y)$$

where S is the upper surface of the plane with equation $2x + 3y + 6z = 12$, which is located in the positive octant.

Problem Set 8 Divergence theorem

Divergence theorem The integral of a vector field normal to a *closed surface* can be expressed in terms of a triple integral of the divergence over the volume V contained in S. In symbols:

$$\oiint_S \mathbf{F} \bullet \mathbf{n}\, d\sigma = \int\int\int_V \nabla \bullet \mathbf{F}\, dxdydz \qquad (5)$$

with the unit normal points outwards from the closed surface.

For Questions 1-3 use the Divergence theorem to evaluate $\oiint_S \mathbf{F} \bullet \mathbf{n}\, d\sigma$.

1.

$$\mathbf{F}(x, y, z) = (x \sin y, \cos 2x, y^2 - z \sin y)$$

over the surface of the sphere S with equation $x^2 + y^2 + (z-2)^2 = 1$.

2. Find $\oiint_S \mathbf{F} \bullet \mathbf{n}\, d\sigma$ where

$$\mathbf{F}(x, y, z) = 2x\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$$

and S is the surface of the unit sphere centred at the origin.

3. Using the Divergence Theorem, find the flux of the vector field

$$\mathbf{F}(x, y, z) = (xy, yz, xz)$$

outward through the surface of the cube cut from the first octant by the planes $x = 1, y = 1, z = 1$.

4. Find $\oiint_S \mathbf{F} \bullet \mathbf{n}\, ds$ through use of the Divergence theorem for the vector field

$$\mathbf{F}(x, y, z) = (4x, -2y^2, z^2)$$

and the surface of the cylinder $S = \{x^2 + y^2 = 4, 0 \leq z \leq 3\}$.

5. Calculate the integral of Question 4 directly.

6. Prove that for any closed surface S and vector field $\mathbf{F}$

$$\oiint_S \nabla \times \mathbf{F}\, d\sigma = 0.$$

7. Find the relationship between the volume contained in the closed surface S and the integral

$$\oiint_S \mathbf{r} \bullet \mathbf{n} \, d\sigma$$

where $\mathbf{r}(x, y, z) = (x, y, z)$.

8. Suppose two scalar fields are related by $\nabla^2 f = g$. Show that

$$\int\int\int_V g \, dxdydz = \oiint_S (\nabla f) \bullet n \, d\sigma.$$

Planar version of the Divergence theorem: let $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$, then for a closed curve C and for the region R contained by C:

$$\oint_C \mathbf{F} \bullet \mathbf{n} ds = \int\int_R \nabla \bullet \mathbf{F} \, dxdy$$

where $\mathbf{n}$ is the outward pointing unit normal $\mathbf{n} = \left(- \frac{dx}{ds}, \frac{dy}{ds} \right)$.

9. Verify that this plane version of the theorem works for the case where

$$\mathbf{F}(x, y) = 2y\mathbf{i} + 5x\mathbf{j}$$

and C is the circle $x^2 + y^2 = 1$.

10. Prove the theorem by showing that the equality follows from Green's theorem.

Problem Set 9 Stokes's theorem

For a smooth vector field $\mathbf{F}$ in three dimensions and an *orientable surface** S with boundary curve C

$$\int\int_S \nabla \times \mathbf{F} \bullet \mathbf{n}\, d\sigma = \oint_C \mathbf{F} \bullet \mathbf{d}r$$

where C is a closed curve parametrized by $\mathbf{r}(t)$ say and where $\mathbf{n}$ is the unit normal to the surface such that $\mathbf{T} = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$, $\mathbf{n}$, and $\mathbf{T} \times \mathbf{n}$ form a *right handed system.*

* *Comment* The *Mobius strip*, which is a (long thin) rectangle with the ends glued together with a half twist is the basic example of a non-orientable surface. The strip has only one side and one edge and in consequence a continuous vector field cannot be applied to it.

1. Use Stokes's theorem to evaluate

$$\oint_C \mathbf{F} \bullet d\mathbf{r}$$

where C is the triangle with vertices $(1,0,0), (0,1,0), (0,0,1)$ traversed in that order and $\mathbf{F}(x,y,z) = (z^2, y^2, x)$.

2. Let C be the curve which begins at $(0,0,0)$, passes as a straight line to $(0,0,1)$, passes as a quarter circle in the yz-plane to $(0,1,0)$ and then as a line segment back to the origin.

Calculate directly the integral $\oint_C \mathbf{F} \bullet d\mathbf{r}$ where $\mathbf{F}(x,y,z) = (y,z,x)$.

3. Re-work Question 2 via the Stokes theorem.

4. Let C be the circle where the cone $x^2 + y^2 = 1$ meets the plane $z = 1$, oriented in the anti-clockwise direction when viewed from the z-axis looking toward the origin. Let

$$\mathbf{F}(x,y,z) = (\sin x - \frac{y^3}{3}, \cos y + \frac{x^3}{3}, xyz).$$

Use Stokes's theorem to evaluate $\oint_C \mathbf{F}(x,y,z) \bullet d\mathbf{r}$.

5. Use Stokes's theorem to prove that any *irrotational vector field* (one with zero curl) has the property that $\oint_C \mathbf{F} \bullet d\mathbf{r} = 0$ for any simple closed curve C.

6. Suppose that S is a flat surface lying in the xy-plane, so that $z = 0$ and $\mathbf{n} = (0,0,1)$. Suppose we have a vector field given by $\mathbf{u}(x,y) = (P(x,y), Q(x,y), 0)$. Show that Stokes's theorem in this case reduces to Green's Theorem in the plane. (And so Stokes's theorem generalises that of Green.)

Maxwell's equations describe the relationship between the *electric field strength* $\mathbf{E}$ and the *magnetic field strength* $\mathbf{B}$:

$$\nabla \bullet \mathbf{B} = 0, \;\; \nabla \bullet \mathbf{E} = \frac{\rho}{\varepsilon_0};$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \;\; \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

where ε_0 and μ_0 are positive constants, ρ denotes electric charge density, t denotes times and $\mathbf{J}$ is another associated vector field known as the *total current density.*

7. Use the Divergence theorem to deduce *Gauss's law of electric fields*, which says that

$$\oiint_S \mathbf{E} \bullet \mathbf{n} \, d\sigma = \frac{Q}{\varepsilon_0}$$

where Q is the total charge enclosed by S.

8. Use Stokes's theorem to deduce the *Maxwell-Faraday equation* in the form

$$\oint_C \mathbf{E} \bullet d\mathbf{r} = -\frac{\partial}{\partial t} \int \int_S \mathbf{B} \bullet \mathbf{n} \, d\sigma,$$

where C is a curve corresponding to the boundary of the surface S.

9. Use the general form of Maxwell's equations to show that the charge density ρ and the electric current density $\mathbf{J}$ obey the conservation law

$$\frac{\partial \rho}{\partial t} + \nabla \bullet \mathbf{J} = 0.$$

10. Suppose the energy of an electromagnetic wave in a vacuum is given by

$$w = \frac{1}{2} \mathbf{B} \bullet \mathbf{B} + \frac{1}{2c^2} \mathbf{E} \bullet \mathbf{E}$$

where $c^2 = \frac{1}{\mu_0 \varepsilon_0}$. Using Maxwell's equations in a vacuum (i.e. where $\rho = 0$ and $\mathbf{J}=\mathbf{0}$), show that the rate of change of the energy obeys the conservation law

$$\frac{\partial w}{\partial t} + \nabla \bullet \mathbf{P} = 0,$$

where $\mathbf{P} = \mathbf{E} \times \mathbf{B}$ is known as the *Poynting vector.*

Problem Set 10 Frenet-Serret equations for curves in 3D space

Let C be a smooth curve in 3-space parametrized by arc length $\beta(s)$ with $\beta'(s) \neq 0$ for all s. Then $T(s) = \beta'(s)$, is a unit tangent vector to the curve. Let $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$ be two smooth vector functions of time t.

1. Show that
$$(\mathbf{r}_1 \bullet \mathbf{r}_2)' = \mathbf{r}_1' \bullet \mathbf{r}_2 + \mathbf{r}_1 \bullet \mathbf{r}_2'.$$

.

2. Similarly verify that
$$(\mathbf{r}_1 \times \mathbf{r}_2)' = \mathbf{r}_1' \times \mathbf{r}_2 + \mathbf{r}_1 \times \mathbf{r}_2'.$$

3. Show that if $\mathbf{r}(t) \bullet \mathbf{r}(t) = c$, a constant, then $\mathbf{r}'(t) \perp \mathbf{r}(t)$. Hence deduce that $\beta''(s) \perp T(s)$.

Define the *curvature* $k(s) = ||\beta''(s)||$. Write $N(s) = \frac{\beta''(s)}{k(s)} = \frac{T'(s)}{k(s)}$, thereby defining the unit *normal vector* at $\beta(s)$; put $B = T \times N$, the (unit) *binormal vector* at $\beta(s)$. The trio (T, N, B) form the *Frenet frame* for β at s, which is an orthogonal trio of unit vectors at $\beta(s)$.

4. Deduce the *First Frenet Equation*,
$$T'(s) = k(s)N(s).$$

5. Show the *Third Frenet Equation*, which is that $B'(s)$ has the form
$$B'(s) = -\tau(s)N(s)$$
for some function $\tau(s)$ that we shall call the *torsion* of β at s. [Hint: show $B'(s) \perp B(s), T(s)$.]

6. By writing $N = B \times T$ deduce the *Second Frenet Equation*,
$$N'(s) = -k(s)T(s) + \tau(s)B(s).$$

7. Show that the three Frenet equations can be written as a single matrix equation
$$(T', N', B') = M(T, N, B)^T,$$
where M is a suitable 3×3 matrix.

8. Consider the *unit speed helix*, $\beta(s) = (a \cos \frac{ws}{c}, a \sin \frac{ws}{c}, \frac{b}{c}s)$ $(s \geq 0)$. Calculate $T(s)$, and deduce that, for the helix to be traversed at unit speed, $c^2 = a^2w^2 + b^2$.

9. Find $T'(s)$, showing that the curvature $k(s)$ is constant, and thus calculate $N(s)$.

10. Find the binormal vector $B(s)$, its derivative $B'(s)$ and thus find the torsion, $\tau(s)$ again showing it to be constant.

Hints for Problems

Problem Set 1

1. Consider the triangle defined by the vectors $\mathbf{a}$ and $\mathbf{b}$ with common tail so that the third side corresponds to $\mathbf{a} - \mathbf{b}$.

4 & 5. Use the determinant form and one of the properties of determinants.

9. It is sufficient to establish equality in the first component as the second and third components are formed in the corresponding way.

Problem Set 2

1. We may assume that the directional derivative in the direction of any tangent vector to a level surface is 0.

2. Find the direction that maximizes the value of the directional derivative at a given point P.

6. The angle between the surfaces at a common contact point equals the angle between their normals.

8. The equation of the tangent plane takes the form $f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = z - z_0$ or in vector notation $(\nabla f|_{\mathbf{x}_0}) \bullet (\mathbf{x} - \mathbf{x}_0) = z - z_0$.

9. Follow your nose: this follows at once from the linearity of differentiation.

10. And this one follows from the product rule.

Problem Set 3

7. The proof of this and similar identities assumes equality of mixed partial derivatives: $\frac{\partial f^2}{\partial x \partial y} = \frac{\partial f^2}{\partial y \partial x}$.

Problem Set 4

2. Make use also of the identity of Question 9 of Set 3.

4. 1. $\int_C \mathbf{F} \bullet \mathbf{dr} = \int_0^1 \mathbf{F} \bullet \frac{\mathbf{dr}}{dt} dt$.

5. Write $\mathbf{F} = (\phi_x, \phi_y)$ and integrate ϕ_x with respect to x but remember this introduces an arbitary function $f(y)$; differentiate with respect to y and equate

with ϕ_y to find $f(y)$ and hence the potential $\phi(x, y)$.

6. Similar to Question 5 but the initial integration determines ϕ only up an arbitary function $f(y, z)$. Differentiate ϕ with respect to y, compare to the second component function of $\mathbf{F}$, and integrate to determine ϕ up to an arbitrary function $g(z)$ say; repeat the procedure using the final variable z in order to determine the potential up to an integration constant.

7. Again use Question 9 of Set 3.

8. Writing down the integral you discover that it is the integral of $\phi'(t)$; the result then follows from the Fundamental theorem of calculus.

Problem Set 5

6. For the alternative formulation, write $\mathbf{r}''(t) \bullet \mathbf{r}'(t)$ as the derivative of half the square of the length of the velocity vector.

10. Make C as large as possible while ensuring that the Green's integrand is positive in the enclosing region.

Problem Set 6

1. Draw the region of integration before setting up your double integral, perhaps fixing x and integrating first with respect to y.

5. Apply Green's theorem to the given integral and see what you get.

6. Use the parametrization $\mathbf{r}(t) = (a \cos t, b \sin t)$ $(0 \le t \le 2\pi)$.

Problem Set 7

1. Parametrize the conical surface by cylindrical coordinates: $\mathbf{r}(t, z) = (z \cos t, z \sin t, z)$ $0 \le t \le 2\pi$, $0 \le z \le 1$.

5. Write $y = g(x, z)$ on the surface (in this case, $g(x, z) = g(x)$ only) and then take $h(x, y, z) = y - g(x, z)$; remember to write the integrand $f(x, y, z)$ in the form $f(x, g(x, z), z)$ and integrate on the projection of the surface onto the xz-plane.

6. Parametrize the surface using $x = r \cos t$,$y = r \sin t$ and $z = \frac{h}{R} r$. for $0 \le t \le 2\pi$ and $0 \le r \le R$.

Problem Set 8

5. This integral is over the entire surface of the cylinder, including the end discs (so the Divergence theorem applies). This makes the calculation relatively long.

9. $\mathbf{n} = (\cos t, \sin t)$, substitute accordingly.

10. We may write $\mathbf{n}\, ds = (dy, -dx)$ and substitution leads to the result via Green's theorem.

Problem Set 9

6. Take x and y as the parameters, remembering that $z = 0$ and interpret the Stokes theorem in this context.

10. Apply the identity $\nabla \bullet (\mathbf{f} \times \mathbf{g}) = g \bullet (\nabla \times \mathbf{f}) - \mathbf{f} \bullet (\nabla \times \mathbf{g})$ to $\nabla \bullet \mathbf{P}$.

Problem Set 10

5. $B(s) \bullet B(s) = 1$.

Answers to the Problems

Problem Set 1

10. $(\mathbf{a} \bullet \mathbf{b})\mathbf{c} = (\mathbf{b} \bullet \mathbf{c})\mathbf{a}$.

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Problem Set 2

3. $(1, 2, -4e^2)$. 4. $\frac{37}{3}$. 5. $\pm\frac{1}{3}(2, 2, -1)$ 6. $0 \cdot 95$ radians 7. $(\frac{5}{169}, -\frac{8}{169})$. 8. $\frac{5}{169}(x-2) - \frac{8}{169}(y-3) = z - \frac{2}{9}$.

Problem Set 3

$3.1 + yz$. 4. $-x\mathbf{i} + xy\mathbf{j} + z(1-x)\mathbf{k}$. 5. $\frac{1}{x} + \frac{x}{x^2+z^2} + xze^{xyz}$. 6. $-xye^{xyz}\mathbf{i} + \frac{z}{x^2+z^2}\mathbf{j} + yze^{xyz}\mathbf{k}$. 7. $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

Problem Set 4

4. $3\frac{31}{60}$. 5. $\phi(x, y) = \frac{1}{3}x^3 + xy^2 + y + c$. 6. $\frac{1}{2}x^2 + 2xy + 4xz - \frac{3}{2}y^2 - zy + z^2 + c$. 7. $\text{curl}(\mathbf{u}) = (0, -xy, 1 - xz) \neq \mathbf{0}$and hence $\mathbf{u}$ is not conservative. 9 & 10. $6\frac{1}{3}$.

Problem Set 5

1. $e - \frac{1}{e^8} - \frac{4}{3}$. 2. $z \sin x + x \ln z + \frac{y^3}{3} - \frac{4}{3}$. 3. -5π. 5. -1. 7. $\left(-\frac{9}{10}, -\frac{2}{3}, \frac{7}{5}\right)$. 8. -5π. 9. -24π . 10. The unit circle centred at the origin, $\frac{\pi}{2}$.

Problem Set 6

1 & 2. $\frac{1}{12}$. 3 & 4. -8π. 6 πab. 7. $A = \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1})]$.8. 8. 9. 2π. 10. 0.

Problem Set 7

1 & 4. $\frac{\sqrt{2}\pi}{4}$. 5. $3\sqrt{2}$ 7. $\frac{3\pi}{2}\left(\sqrt{2} + \ln(1+\sqrt{2})\right)$. 8. π. 10. 24.

Problem Set 8

1. 0. 2. 0. 3. $\frac{3}{2}$. 4 & 5. 84π. 7. $3V$. 9. 0.

Problem Set 9

1. $-\frac{1}{6}$ 2&3 $\frac{\pi}{4}$. 4. $\frac{\pi}{2}$.

Problem Set 10

7. $\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$.8. $T(s) = \beta'(s) = (-\frac{aw}{c}\sin\frac{ws}{c}, \frac{aw}{c}\cos\frac{ws}{c}, \frac{b}{c})$. 9. $T'(s) = (-\frac{aw^2}{c^2}\cos\frac{ws}{c}, -\frac{aw^2}{c^2}\sin\frac{ws}{c}, 0)$; hence $||T'(s)|| = \frac{aw^2}{c^2} = k(s)$; $N(s) = \frac{T'(s)}{k} = -(\cos\frac{ws}{c}, \sin\frac{ws}{c}, 0)$. 10. $B(s) = (\frac{bw}{c^2}\cos\frac{ws}{c}, \frac{bw}{c^2}\sin\frac{ws}{c}, 0)$; $B'(s) = -\tau(s)N(s)$; $\tau = \frac{bw}{c^2} = \frac{bw}{a^2w^2+b^2}$