Mathematics 104 Numbers & Discrete Mathematics

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This module delves into many topics in which numbers and their properties feature. Some of the questions highlight novel aspects of arithmetic but some very fundamental topics are introduced. In particular, in Set 4 we see the *Euclidean algorithm*, applied to both whole numbers and fractions. This ancient piece of mathematics is fundamental to modern cryptography (see MA202 Set 3) and is generalized in certain forms of abstract algebra known as *ring theory*. Everyone knows how important are the *prime numbers* and *the fundamental theorem of arithmetic*, which says that every natural number has a unique prime factorization, and so prime numbers feature in several problem sets, particularly Set 5. Special integers, such as *perfect numbers* are met as well.

In Sets 6 and 7 we make an excursion into *logic* and *set theory*, with a set of logic puzzles, a few examples on *truth tables* and formal logical argument, and some exercises in *boolean set notation* and the *set laws* including *De Morgan's law*. In Set 8 there are exercises in *induction* and *recursion* with the famous *Fibonacci numbers* being introduced for the first time. The standard *arithmetic* and *geometric series* turn up as well, as they do frequently throughout mathematics.

Another topic that we first encounter in this module is *modular arithmetic* and the *congruence notation* $a \equiv b \pmod{m}$ to mean that $a - b$ is some multiple of the *modulus* m.

Set 10 does introduce a fresh topic as well, that of the *continued fraction* representation, which is especially interesting when applied to square roots of integers as the representations that we obtain are always recurring, which of course is not true for the decimal expansions of a number such as $\sqrt{2}$, which is *irrational*.

Problem Set 1 Numbers

1. Express the recurring decimal $0 \cdot \dot{6}\dot{3}$ as a fraction cancelled to its lowest terms.

2. What is the number, the binary representation of which is

$$11010110?$$

3. Express 100 in binary.

4. Convert the recurring ternary fraction $(0 \cdot \dot{2}\dot{0})_3$ to a base 10 fraction.

5. Write $\frac{9}{20}$ as the sum of two unit fractions.

6. Show that 1729 is the sum of two cubes in two different ways.

7. A survey reveals that 76.8% of students think that university fees are too high. What is the smallest possible size for the sample?

8. A positive integer is called *perfect* if it is equal to the sum of its factors that are less than itself: for example 6 is perfect as $6 = 1 + 2 + 3$.

A famous theorem of Euclid (ca. 300 BC) is that *if p and $2^p - 1$* are both prime then $2^{p-1}(2^p - 1)$ is perfect. Find the first three perfect numbers yielded by Euclid's formula.

9. A pair of numbers is called *amicable* if the sum of the factors of each member of the pair (not including the number itself) is equal to the other. Show that 220 is one half of an amicable pair.

10. If different letters stand for different digits, find the meaning of the sum

$$\text{FRED}+\text{EATS}=\text{ADDER}.$$

Problem Set 2 Numbers II

1. How many digits are there in the binary representation of one million?

2. Write $\frac{1}{4}$ in binary.

3. Find a set of seven positive integers such that any number from 1 to 100 is the sum of some subset.

4. A *pythagorean triple* is an ordered triple of positive integers (a, b, c) such that $a^2 + b^2 = c^2$. Let $p > q$ be positive integers. A pythagorean triple $(a, p^2 - q^2, p^2 + q^2)$ can always be formed. Express a in terms of p and q.

5. Find a right triangle with hypotenuse 17 and sides of integral length.

6. Write $\frac{6}{13}$ as the sum of reciprocals of three distinct positive integers.

7. Write down the 5th *Farey sequence*, which is the ascending list of fractions between 0 and 1 with denominators no more than 5.

8. The USA has coins with values 1c, 5c, 10c, 25c and 50c. Find a way of making a dollar using exactly 50 coins.

9. Find a second solution to Question 8.

10. *Ancient Arabian Problem* A hunter was hunting alone and he was running short of food. Luckily he met two shepherds, one of whom had three small loaves and the other five. When the hunter asked them for food, they agreed to divide the loaves equally among the three of them. The hunter thanked them for the meal and paid the shepherds eight piasters. How should the shepherds divide the money?

Problem Set 3 Numbers III

1. Express 146 in *ternary*, that is, to the base 3.

2. The vertices of a cube are labelled with the numbers $1-8$ so that each face sums to the same number. What is that number and find such a labelling of the cube.

3. Repeat Question 2 replacing 'vertices' by 'edges'.

4. Find the value of the infinite continued fraction:

$$\phi = 1 + \cfrac{1}{1 + \frac{1}{1+\frac{1}{1+\cdots}}}.$$

5. How many numbers are there up to and including $1,000$ that are not multiples of either 2 or 3?

6. Find the smallest integer divisible by all the numbers from 1 to 10.

7. What are the last two digits of 7^{355}?

8. What are the last two digits of $0! + 5! + 10! + \cdots + 100!$?

9. What is the largest possible value of a collection of silver and copper coins from which it is NOT possible to make up exactly £1.00? (Available coin types are 1p, 2p, 5p, 10p, 20p, 50p.)

10. Given that different letters stand for different numbers, decipher the message:

SEND+MORE=MONEY.

Problem Set 4 Euclidean algorithm

The *Euclidean algorithm* outputs the *highest common factor* (hcf), also known as the *greatest common divisor* (gcd) of two positive integers a and b, denoted by (a, b), by repeatedly subtracting the smaller number from the larger until the two numbers in hand are identical. For example,

$$(108, 72) \mapsto (72, 36) \mapsto (36, 36) \Rightarrow hcf(108, 72) = 36.$$

1. Find the hcf of 3675 and 2058 using the Euclidean algorithm.

2. Answer Question 1 for the pair 516 and 432.

3. By reversing the calculation of Question 2, express the hcf in the form $516m + 432n$ for some $m, n \in \mathbb{Z}$.

4. Find two integers m and n with $0 \leq m \leq 34$ such that $22m + 35n = 1$.

5. What is the greatest common divisor of $\frac{4}{9}$ and $\frac{8}{15}$?

6. Repeat Question 5 for the least common multiple.

7. Find a general expression for the greatest common divisor of two rationals $(\frac{a}{b}, \frac{c}{d})$.

8. Repeat Question 7 for least common multiple.

9. Show that if both rationals are given in reduced form, the same is true of the output of your formulas for Questions 7 and 8.

10. Show that the Euclidean algorithm will apply to pairs of fractions and use it to answer Question 5.

Problem Set 5 Prime numbers

1. Find the prime factorization of 323.

2. De Bouvelles (1509) said that at least one of the numbers $6n \pm 1$ is always prime for any positive integer n. Prove him wrong by finding the least n for which his claim fails.

3. Show that every prime number, apart from 2 and 3, has the form $6n \pm 1$.

4. *Twin primes* are consecutive odd numbers, such as 29 and 31, which are both prime. The pairs $(11, 13)$ and $(17, 19)$ are 'consecutive' pairs of twin primes in that they are as close as possible as no number ending in 5 (except 5) can be prime. Find the next duo of consecutive pairs of twin primes ending $1, 3, 7, 9$ in this fashion.

5. Show that the arithmetic progression $a + nd$, where $a \geq 2$, $d \geq 1$ and $n = 1, 2, \cdots$ cannot be a *formula for primes*, in other words the sequence contains some composite numbers.

6. Show that the arithmetic progression $1 + nd$ where $d \geq 1$ and $n = 1, 2, \cdots$ cannot be a formula for primes.

7. Find a non-negative integer x such that $x^2 - x + 41$ is not prime.

8. Find a sequence of 100 consecutive numbers, none of which are prime. [Hint: can $n! + k$ where $0 \leq k \leq n$ be prime?]

9. Show that for any odd prime p, $p^2 - 1$ is divisible by 8.

10. What is the greatest common factor of the set of numbers $p^4 - 1$ where p is a prime greater than 5?

Problem Set 6 Logic Puzzles

1. Four children are playing when the window is broken. Alexander says that Barbara did it, Barbara said that Caroline did it while each of Caroline and David say didn't see what happened. Assuming that only the guilty child is not telling the truth, who broke the window?

2. A chocolate bar consists of n identical squares. What is the minimum number of snaps required to break the bar up into single squares?

3. The King Octopus has Octopus servants with 6, 7 and 8 legs. They all tell the truth except for the 7-legged ones, which always lie. You meet four of them. The Blue one says 'All together we have 25 legs', the Red one says, 'No, all together we have 26 legs, but the Green one says, 'No, we collectively have 27 legs', while the Golden Octopus says, 'I count 28 legs in all'. How many legs has each octopus?

4. *(Bishop Alcuin's Problem, 8th century York)* A boy has to transport a dog, a goose, and a bag of grain across the river but his boat is only big enough for him and one of his three charges. What is more, if left together, the goose will eat the grain and the dog will attack the goose. How can he do it?

5. Two friends wish to share 8 litres of wine that fills a big pitcher. They have two empty vessels of capacities 3 and 5 litres respectively. Show that they may use what they have to share the wine equally.

6. Jack is looking at Anne but Anne is looking at George. Jack is married but George is not. Can we say whether or not a married person is looking at an unmarried person?

7. We have one pint of juice. I drink half and hand it to you, and you drink half of what remains. We continue in this fashion with each of us drinking half of what remains on each turn. How much do I drink?

8. In my school days when my class was lined up by height, I was right in the middle. Must to my chagrin, the girl that I secretly fancied was taller than me and stood 15th in line, from shortest to tallest, while the boy I knew she secretly fancied was in 26th spot. I also noticed that this was enough information to infer how many of us were in the class. How many were there?

9. You are trapped in the centre of a flat circular area of radius $n \geq 1$ by an evil maniacal tyrant. You have to walk out by taking steps of size 1. However, for each step, you can only choose the direction to walk in, and the tyrant will dictate if you move in the positive or negative direction. (That is, you pick the line of your walk but the tyrant tells you which direction on the line to take your step. After taking the step, you pick another line and he tells you which way to go, ... and so on.)

What is the minimum number of steps you need to take to guarantee that you can reach the circumference?

10. Three friends A, B, and C play table tennis in pairs all afternoon. After each game, the loser sits the next game out and in the end they play 10,15 and 17 games each respectively. Who lost the second game?

Problem Set 7 Sets and Logic

1. Use the laws of sets to simplify the expression $(A \cap B) \cup (B' \cap A)$, (where B' denotes the complement of the set B) and write down the dual statement that arises by interchanging $\cap$ and $\cup$ throughout.

2. Simplify the expression of sets $(A' \cap B)' \cap (A \cup B)$.

3. How many sets of people of size at least two can be formed from a group of twelve?

4. In a school of 100 students, 50 take French, 40 take German and 20 take both languages. How many take neither language?

5. *Generalized distributive law for sets*: taking the $n = 2$ case as given, prove by induction that for any sets A and $B_1, B_2, \cdots, B_n$ $(n \geq 2)$:

$$A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n).$$

6. Show that $p \wedge (\sim p)$ is a tautology, which is to say is always true no matter what truth value is assigned to p.

7. Use a truth table to show the logical equivalence:

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r).$$

8. We write $p \rightarrow q$ to mean that if p is true then q is true. The contrapositive: show the three-way logical equivalence

$$p \rightarrow q \equiv (\sim q) \rightarrow (\sim p) \equiv \sim (p \wedge (\sim q)).$$

9. Implication is not equivalent to converse. Show that $(p \rightarrow q) \not\equiv (q \rightarrow p)$.

10. Test the validity of the following argument by use of a truth table:

$$p \vee q, \; p \rightarrow r, \; q \rightarrow r, \; \therefore r.$$

Problem Set 8 Induction and Recursion

Prove the claims in Questions 1-4 by induction on n.

1.

$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}.$$

2.

$$1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2 = (1+2+\cdots+n)^2.$$

3. $n(n^2+5)$ is divisible by 6 for all $n = 1, 2, \cdots$.

4. The summation formula for geometric progressions:

$$a + ar + ar^2 + \cdots + ar^{n-1} = a \cdot \frac{1-r^n}{1-r},\ (n \geq 1).$$

5. The *Fibonacci numbers* are defined by $F_0 = 0$, $F_1 = 1$ and for $n \geq 2$ $F_n = F_{n-1} + F_{n-2}$. Write down the Fibonacci sequence as far as F_{10}.

6. Show by induction that $\sum_{k=0}^{n} F_k = F_{n+2} - 1$.

7. The alternative Fibonacci numbers are defined by $f_0 = 0, f_1 = 1$ and each number is equal to the sum of its two *successors*. Write down the first ten numbers is this sequence.

8. Write down a simple relationship between the sequence of Question 7 and the ordinary Fibonacci numbers and prove it.

9. Consider the sequence of fractions that begins with $a_0 = 0$, $a_1 = 1$ and defined thereafter recursively by the rule:

$$a_n = \frac{a_{n-2} + a_{n-1}}{2}.$$

Write down the the sequence up to and including a_8.

10. Find the limiting value as $n \to \infty$ of the sequence a_n.

Problem Set 9 Numbers IV

1. Show that $17!18!/2$ is a perfect square.

2. A class of 48 students has a ratio of boys to girls of $3:5$. How many male students would needed to be added to the class in order to reverse this ratio?

3. Simplify $\sqrt{3\sqrt{2\sqrt{3\sqrt{2}}}}\cdots$.

4. Find the exact value of $\sqrt{4+\sqrt{4-\sqrt{4+\sqrt{4-}}}}\cdots$.

5. Show that the sum of two squares is never congruent to 3 modulo 4.

6. Show that no integer of the form $8k+7$ is the sum of three squares.

7. Show that the curve with equation $x^2 - 19y^2 = 15$ has no *lattice points*, which is to say points where both co-ordinates are integers.

8. Write down the set of positive integers that consist of all numbers less than 15 that share no common factor (apart from 1) with 15. For a given positive integer n we denote by $\phi(n)$ the number of positive integers less than n with no factor in common with n. It is known that $\phi(n)$ is given by the formula:

$$\phi(n) = n(1-\frac{1}{p_1})(1-\frac{1}{p_2})\cdots$$

where the p_i represent all the distinct prime factors of n. Find $\phi(323)$.

9. Find

$$\lim_{n\to\infty}\sum_{k=1}^{n}|e^{\frac{2\pi ki}{n}} - e^{\frac{2\pi(k-1i)}{n}}|.$$

10. Is $2^{50}+1$ a prime?

Problem Set 10 Numbers IV

1. A *hailstone sequence* is the sequence of numbers that arises by beginning with a particular number n and following the rule: if n is even, divide by 2, but if n is odd replace it by $3n-1$. Find the hailstone sequence that begins with 7.

2. The *aliquot function* $a(n)$ is the sum of all the factors of n less than itself. Its value is given by the product of all the terms $\frac{p^{k+1}-1}{p-1}$, where p ranges over all the prime factors of n. Find $a(276)$.

The *Stirling Number of the Second Kind*, $S(n, r)$ is the number of ways of partitioning a set of n members into r blocks (with no block empty, and the order of blocks and the order within blocks is immaterial).

3. Find the values of $S(n, 1)$ and $S(n, 2)$.

4. Continuing with Question 3, find the values of $S(n, n)$ and $S(n, n-1)$.

5. Find x given that $x! = \frac{7!!}{7!}$.

Continued fractions are a tool used in number theory to express a real number r in a unique fashion without reference to any base. They are replete with many special properties and are finite exactly when r is rational and repeating exactly when r is the root of a quadratic with integer coefficients. Here we consider the simple form using integers $a_0, a_1, \cdots$.

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 \ldots}}}$$

which we shall denote as $[a_0; a_1, a_2, a_3, \cdots]$. The formula for this expansion begins with $a_0 = \lfloor r \rfloor$, the integer part of r and in general $a_n = \lfloor r_n \rfloor$, where r_n is defined recursively as $r_0 = r$ and $r_n = \frac{1}{r_{n-1} - a_{n-1}}$. Find the continued fraction representation of each of the following numbers.

6. $\frac{45}{16}$.

7. $\frac{16}{45}$.

8. $\sqrt{2}$.

9. $\sqrt{7}$.

10. The Golden ratio $\phi = \frac{1+\sqrt{5}}{2}$.

Hints for Problems

Problem Set 1

1. Call the given number a, multiply by 100 and find out the value of $100a - a$.

4. Follow the method of Question 1 but use base 3.

5. Repeatedly subtract the largest unit fraction available and the decomposition soon appears.

7. Consider the fraction $\frac{a}{b}$ where b is the sample size and a is the number that agree with the proposal.

Problem Set 2

1. Find the least power of 2 that exceeds your target.

3. Think binary again!

5. Subtract the largest reciprocal you can and repeat until you get an answer.

Problem Set 3

4. Use the continued fraction to infer a quadratic equation with ϕ as one of its roots.

6. Find a small power of 7 that is close to a multiple of $m = 100$ and use *modular arithmetic:* $a \equiv b \pmod{m}$ if $a = b + km$ for some multiple of m. All arithmetic operations (excpet division) can be freely applied to modular equations.

Problem Set 4

3. Begin with the second to last line of the equation and work backwards.

7 & 8. These are harder question. You need to find a formula, in terms of the numerators and denominators of the given numbers and their greatest common divisors and least common multiples. Then you should prove they are

correct. Any proof will use standard properties of divisors. In particular, if p is a prime and $p|ab$ (read 'p is a factor of ab') the $p|a$ or $p|b$. This is known as

Problem Set 5

3. Consider each number modulo 6.

10. Factorize and consider the terms modulo 2, 3, and 5 to find necessary factors of $p^4 - 1$.

Problem Set 6

9. Try to nullify the tyrant's power.

10. Look to totals and the number of games played by B against C.

Problem Set 7

This problem set requires knowledge of the *Laws of sets* under *intersection* $\cap$ and *union* $\cup$ and *complement* $(\cdot)'$; the corresponding *Laws of logic* describe the behaviour of *conjunction* $\wedge$ ('and' symbol) *disjunction* $\vee$ ('or' symbol) and *negation* ($\sim$, 'not'). The last group of questions involving drawing up *truth tables* and drawing conclusions from them.

3. The order of the *power set* (the set of all subsets) of a set of order n is 2^n.

10. For the argument to be valid the conclusion has to be true in the *critical rows*, which are the rows where *all* premises are true.

Problem Set 8

10. Look for a solution of the form $A\alpha^n + B\beta^n$.

Problem Set 9

3 & 4 Through squaring, make the number the subject of a simple equation.

5 & 6 Look through all the possibilities for squares modulo 4 and 8 respectively.

7. Again, look at what is possible modulo 4.

9. Think geometrically.

10. Look for a general type of factorization of $a^n + 1$ where n has an odd factor.

Problem Set 10

5. Cancel down the right hand side.

Answers to the Problems

Problem Set 1

1. $\frac{7}{11}$. 2.214. 3. 1100100_2. 4. $\frac{3}{4}$. 5. $\frac{1}{3}+\frac{1}{10}+\frac{1}{60}=\frac{1}{4}+\frac{1}{5}$. 6. $1^3+12^3=9^3+10^3$. 7. 125. 8. $6, 28, 496$. 9. 284 . 10. $7,052+5,198=12,250$; $5,072+7,198=12,270$.

Problem Set 2

1.20. 2. $(0 \cdot 01)_2$. 3. $\{1,2,4,8,16,32,64\}$ 4. $2pq$. 5. $(8,15,17)$. 6. $\frac{1}{3}+\frac{1}{8}+\frac{1}{312}$. 7. $\{\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{2}{3},\frac{3}{4},\frac{4}{5}\}$. 8. $(40\times 1)+(8\times 5)+(2\times 10)=100$. 9. $(45\times 1)+(2\times 5)+(2\times 10)+(1\times 25)=100$. 10. $1:7$.

Problem Set 3

1. $(12102)_3$. 2. 18. 3. 26. 4. $\frac{1+\sqrt{5}}{2}$. 5. 333. 6. 43. 7. 21. 8. 2520 . 9. £1.43. 10. $9,567+1,085=10,652$.

Problem Set 4

1. 147 . 2. 12. 3. $m=-5$, $n=6$. 4. $m=8$, $n=-5$. 5. $\frac{4}{45}$. 6. $\frac{8}{3}$. 7. $\frac{[b,d]}{(a,c)}$. 8. $\frac{[a,c]}{(b,d)}$. 10. $\frac{4}{45}$.

Problem Set 5

1. 17×19 . 2. $n=20$. 4. $101, 103, 107, 109$, 7. 41 . 8. $101!+2, \cdots, 101!+100$. 10. 240 .

Problem Set 6

1. Barbara. 2. $n-1$. 3. All have 7 legs except Green has 6. 6. Yes. 7.$\frac{2}{3}$. 8. 27. 9. n^2. 10. A.

Problem Set 7

1. A, $(A \cup B) \cap (B' \cup A) = A$. 2. A. 3. 4,083. 4. 30.

Problem Set 8

5. $0, 1, 1, 2, 3, 5, 8, 13, 21, 34$. 7. $0, 1, -1, 2, -3, 5, -8, 13, -21, 34$. 8. $f_n = (-1)^n F_n$. 9. $0, 1, \frac{1}{2}, \frac{3}{4}, \frac{11}{16}, \frac{21}{32}, \frac{43}{64}, \frac{85}{128}$. 10. $\frac{2}{3}\left(1 + \frac{(-1)^{n+1}}{2^n}\right)$, $n = 0, 1, 2, \cdots$.

Problem Set 9

2. 32. 3.$\sqrt[3]{18}$. 4. $\frac{1+\sqrt{13}}{2} \approx 2 \cdot 3028$. 8. $\{1, 2, 4, 7, 8, 11, 13, 14\}$, 288. 9. 2π . 10. No.

Problem Set 10

2. 672. 3. $S(n,1) = 1$, $S(n,2) = 2^{n-1} - 1$. 4. $S(n,n) = 1$, $S(n,n-1) = \frac{1}{2}n(n-1)$. 5. 5039. 6. $[2; 1.4.13]$. 7. $[0; 2, 1, 4, 13]$. 8. $[1; \overline{2}]$. 9. $[2; \overline{1, 1, 1, 4}]$. 10. $[1, \overline{1}]$.