Mathematics 103 Calculus I

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This is the first of many calculus based modules. Basic calculus is assumed, by which is meant knowledge of differentiation and integration of the elementary functions, including the *trigonometric*, *exponential,* and *logarithmic* functions together with the rules that allow you to calculate these. In particular, the *product, quotient,* and *chain rules of differentiation* and the corresponding integration techniques, which are *integration by parts* and *integration using substitution* arise frequently. (Since the quotient rule is a consequence of the chain rule and the product rule, there is little use in learning a specific integration technique based on reversing the quotient rule.) *Inverse trigonometric functions* are dealt with similarly. The *Fundamental theorem of calculus*, which states that differentiation and integration are inverse processes, also emerges in the problem sets.

Standard applications are also met, such as *maximization*, *related rates* problems based on the chain rule, and *area under curves* and *volumes of revolution.*

In Sets 6 and 7 we introduce the *hyperbolic functions* and study their properties, identities, and their contribution to calculus.

Partial fractions as a method used in integrating *rational functions* (quotients of polynomials) also features. In Set 9 we introduce the so-called *t-substitution*, which can be used to turn integrals involving quotients of trigonometric functions into integrals of rational functions.

Problem Set 1 Differentiation

1. Find the derivative of $y = \sec^2(x^2)$.

2. By first taking logarithms of both sides, find the derivative of

$$y = \frac{x^2(7x-14)^{\frac{1}{3}}}{(1+x^2)^4}.$$

3. Find the equation of the tangent to the graph of the function:

$$y = x^3 - 5x^2 + x + 6$$

at the point where $x = 1$, giving your answer in the form $ax + by + c = 0$.

4. Use implicit differentiation to find an expression for $\frac{dy}{dx}$ given that

$$5y^2 + \sin y = x^2.$$

5. Find the slope of the tangent line at the point $(4, 0)$ to the graph of

$$7y^4 + x^3y + x = 4.$$

6. Find $\frac{d}{dx}(\text{cosec }(2\cot 3x))$.

7. Find the derivative of

$$y = \cos^{-1} x - \frac{1}{\sqrt{1-x^2}} + \frac{1}{2^x}.$$

(Note that $\cos^{-1} x$ stands for the inverse cosine function, and not for $\frac{1}{\cos x}$.)

8. Find the derivative of $y = \ln(\frac{\cos x}{1-\sin x})$, simplifying your answer as far as possible.

9. Find the co-ordinates of the point on the graph of $f(x) = \frac{\ln x}{x}$ where the maximum occurs.

10. Find necessary and sufficient conditions on a, b, c and d to ensure that

$$f(x) = \frac{ax+b}{cx+d}$$

is a constant function. [Hint: what is the derivative of a constant function?]

Problem Set 2 Integration

1. By integrating by parts find
$$\int x \cos x \, dx.$$

2. Evaluate
$$\int_0^{\frac{\pi}{4}} \tan x \, dx,$$
expressing your answer in the form $a \log b$, where a and b are rational numbers.

3. Find
$$\int \frac{1-x}{1+x} \, dx.$$

4. By using a suitable trigonometric identity find $\int \cos 6x \cos 11x \, dx$.

5. Find $\int \sec^6 \theta \, d\theta$ by substituting $u = \tan \theta$.

6. Find
$$\int_1^2 x^2 \sqrt{x-1} \, dx.$$
(Give the answer as a non-decimal fraction.)

7. By use of partial fractions find
$$\int \frac{dx}{x^2 + x - 2}.$$

8. By use of a suitable substitution find $\int \frac{dx}{1+e^x}$.

9. Use integration by parts to find $\int \ln x \, dx$.

10. Evaluate
$$\int_0^{\frac{\pi}{2}} |\cos 2x| \, dx.$$
(Perhaps think of it as an area: integrating a positive function like this must give a positive result.)

Problem Set 3 Functions

1. Find the function $f(x)$ that satisfies the equation

$$f(2x+3) = x^2 + 1.$$

2. Repeat Question 1 for *hyperbolic cosine function*:

$$f(e^x) = \cosh x = \frac{e^x + e^{-x}}{2}.$$

3. Show that any function over the real line can be written uniquely as the sum of an even function and an odd function. What is this decomposition in the case of the exponential function?

4. Find a linear solution $f(x)$ to

$$(f \circ f)(x) = 2x + 1.$$

5. Determine all linear functions $f(x) = ax + b$ that are *self-inverse*, that is to say have the property that $(f \circ f)(x) = x$.

6. Given that

$$f(f(x)) = x^2 - x + 1,$$

find $f(0)$.

7. Find the equation of the inverse of the function $y = \frac{ax+b}{cx+d}$ $(ad \neq bc)$.

8. Sketch the graph of $y = \arccos(\cos x)$ for $-2\pi \leq x \leq 2\pi$.

9. Determine the natural domain of the function:

$$f(x) = \frac{\ln(2-x)}{\sin x + \cos x}.$$

10. Repeat Question 9 for $f(x) = \ln \ln(\sin x)$.

Problem Set 4 Differentiation II

1. By first taking logarithms of both sides, or otherwise, find the derivative of $y = a^{-2x}$, $(a > 0)$.

2. Find from first principles (that is to say, directly from the definition) the derivative of $y = x^n$.

3. Find the derivative of $y = \arctan\left(\frac{1}{1+x^2}\right)$.

4. Find the derivative of $y = \sec^{-1}(x)$, the inverse secant function.

5. Find the co-ordinates of all the points on the graph of the standard normal pdf

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

where the second derivative is equal to zero.

6. Find the equation of the normal to the curve $y = cosec\theta$ at the point where $\theta = \frac{\pi}{6}$.

7. Suppose that $w = \tan x$ and that $x = 4t^3 + t$. Find $\frac{dw}{dt}$.

8. Find $\tan\theta$, where θ is the angle made by the x-axis and the tangent of the curve

$$y = \sin 2x + \cos 4x,$$

when $x = \frac{\pi}{8}$.

9. A right circular cylinder S is inscribed in a right circular cone C, with the bases of S and C concentric and such that the volume of S is as large as possible. Find the ratio of the volume of S to that of C.

10. Find $\frac{dy}{dx}$ given that

$$y = \int_0^x \sin(t^2)\, dt.$$

Problem Set 5 Integration II

1. Use the identity $\cos 2\theta = 2\cos^2\theta - 1$ to find

$$\int \frac{dx}{1+\cos x}.$$

2. Find

$$\int \frac{dx}{x \ln x}.$$

3. Find a *primitive* (that is to say, an antiderivative) for $\frac{1}{x^2+6x+18}$.

4. By integrating by parts twice, find

$$\int e^x \cos x \, dx.$$

5. Evaluate

$$\int_0^1 a^{-2x} dx.$$

6. Use integration by parts to find a primitive for $\arcsin x$.

7. Find

$$\int \sin^4 x \cos^5 x \, dx$$

by substituting $u = \sin x$.

8. Find $\int \sin^2 2x \, dx$ by invoking an appropriate trigonometric identity.

9. Find the volume of the solid of revolution formed by rotating the curve $y = \frac{1}{x}$ from $x = 1$ to ∞ about the x-axis.

10. Find

$$\int \frac{dx}{x^2\sqrt{4-x^2}}$$

by using the substitution $x = 2\sin\theta$ $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$, expressing your answer as an algebraic function of x.

Problem Set 6 Hyperbolic functions I

This problem set is about the *hyperbolic functions*: $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\tanh x = \frac{\sinh x}{\cosh x}$.

1. Show that $\cosh^2 x - \sinh^2 x = 1$.

2. Show that $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

3. And similarly that$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$.

4. $$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}.$$

5. $\cosh 2x = 2\cosh^2 x - 1$ and $\sinh 2x = 2 \sinh x \cosh x$.

6. $\sinh x + \sinh y = 2 \sinh \frac{x+y}{2} \cosh \frac{x+y}{2}$.

7. $$\sinh\left(\frac{x}{2}\right) = \operatorname{sgn}(x)\sqrt{\tfrac{\cosh x - 1}{2}}.$$

In Questions 8-10, establish the given formula for each of the inverse hyperbolic functions.

8. $$\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1}).$$

9. $$\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1});\ \ x \geq 1.$$

10. $$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),\ \ |x| < 1.$$

Problem Set 7 Hyperbolic functions II

1. Find the derivatives of $\cosh x$ and $\sinh x$.

2. Show that the derivative of $\tanh x$ is $\text{sech}^2 x = 1/\cosh^2 x$.

3. Show that $1 - \tanh^2 x = \text{sech}^2 x$.

Evaluate each of the following integrals by using an appropriate hyperbolic substitution, taking $a \geq 0$ throughout.

4.
$$\int \frac{dx}{\sqrt{a^2 + x^2}}.$$

5.
$$\int \frac{dx}{\sqrt{x^2 - a^2}}.$$

6.
$$\int \frac{dx}{a^2 - x^2}, \text{ where } x^2 > a^2.$$

7. Find the Taylor series about 0 for each of $\sinh x$ and $\cosh x$.

8. Show that $\sinh x = -i \sin(ix)$ and $\cosh x = \cos(ix)$.

9. Assuming the double angle formula for cosine, show that

$$\cos(x + iy) = \cos x \cosh y - i \sinh x \sinh y.$$

10. Similarly show that

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y.$$

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Problem Set 8 Differentation III

1. Find the derivative of $y = \cot(\sin x)$.

2. Find the derivative of $y = \cot^{-1} x$, the inverse cotangent function.

3. Find the derivative of $y = x^x$.

4. We are given a straight fence 100m in length, and we are to form a rectangular enclosure using an additional 200 metres of fencing. What should be the dimensions of the enclosure to have maximum area?

5. Find the dimmest point between two candles of brightness a and b respectively that are separated by a distance d.

6. A water tank, in the shape of an inverted cone has depth 5m and radius 2m. When the depth of water is 4m, it leaks out at a rate of $\frac{1}{12}m^3$/minute. How fast is the level of water in the tank dropping at that time?

7. Find y' given that

$$y = \int_0^{\sqrt{x}} \sin(t^2)\,dt.$$

8. Find from first principles the derivative of $y = \sin x$.

9. Find the equations of the tangents to the curve $y = x^2$ that pass through the point $(2, 0)$.

10. What is the derivative of $y = \log_x a$?

Problem Set 9 Integration III

The first five questions involve the substitution $t = \tan\left(\frac{\theta}{2}\right)$, together with the trigonometric identities $\sec^2 x = 1 + \tan^2 x$, and $\cos 2x = \cos^2 x - \sin^2 x$.

1. Express $\cos\theta$ as an algebraic function of t.

2. Express $\sin\theta$ as an algebraic function of t.

3. Similarly express the differential $d\theta$ in terms of t and dt.

4. Hence find $\int \operatorname{cosec}\theta \, d\theta$.

5. Find $\int \sec\theta \, d\theta$.

6. Express $\frac{1}{x^2(1-x)}$ in the form

$$\frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-1},$$

and hence find $\int \frac{dx}{x^2(1-x)}$.

7. Find the volume of the solid generated when the region R in the first quadrant enclosed between $y = x$ and $y = x^2$ is revolved around the y-axis.

8. Find $\int \tan^2 x \, dx$ by use of an appropriate trigonometric identity.

9. Evaluate

$$\int_0^{2\pi} \sin mx \cos nx \, dx,$$

where m, n are distinct non-negative integers.

10. Find the integral of the inverse tan function.

Problem Set 10 Integration IV

1. Integrate by parts twice to find
$$\int x^2 e^{-x}\, dx.$$

2. Integrate by parts twice to find $\int e^x \sin x \, dx$ and hence also find
$$\int \frac{\sin x}{e^x}\, dx.$$

3. Evaluate $\int_{-1}^{1} \sin^{2n+1} x \, dx$, where n is a non-negative integer.

4. Find $\int \frac{x^2}{1+x^2}\, dx$.

5. Evaluate the improper integral:
$$\int_0^1 \frac{dx}{\sqrt{1-x}}.$$

6. Find
$$\int \frac{dx}{\sin x + \cos x}.$$

7. Find
$$\int \frac{\sin x}{\cos(1+x)}\, dx.$$

8. Find $\int \sqrt{a^2 - x^2}\, dx$ by substituting $x = a \sin t$.

9. Repeat Question 8 using the substitution $x = a \cos t$.

10. Why do the answers to Questions 8 and 9 not contradict each other?

Hints for Problems

Problem Set 1

1. You need to know that $(\sec x)' = \sec x \tan x$ and then apply the Chain Rule, very carefully, twice.

2. This may be a new technique for you. The idea is to take logs of both sides *before* you differentiate in order to break up the expression into bits that are easy to deal with. The LHS becomes $\log y$ which, when differentiated gives $\frac{y'}{y}$ (by the Chain Rule). You can then make y' the subject of the formula and re-write y using the original expression to get y' entirely in terms of x.

4. The derivative of $y^2(x)$ with respect to x leads, by the Chain Rule, to $2yy'$.

6. Use the Chain Rule and the facts that the derivative of cosec x is $-\cot x(\text{cosec}x)$ and that similarly $(\cot x)' = -\text{cosec}^2 x$.

7. You need to know the derivative of the inverse cosine function and how to find the derivative of 2^{-x}; logarithmic differentiation can help with the latter. Alternatively, we can write a^x as $e^{(\ln a)x}$.

10. Remember that a function $f(x)$ is constant iff its derivative $f'(x)$ is identically zero, that is $f'(x) = 0$ for ALL values of x.

Problem Set 2

2. Write tan$x$ as sin$x$/cos$x$ and substitute $u = \cos\ x$, remembering to change the limits as well.

3. When the degree of the denominator is less than or equal to that of the numerator, divide bottom into top before proceeding.

4. You need to write the integrand as a sum of trigonometric terms that you can then integrate separately.

8. Substitute $u = 1 + e^x$ (or $u = e^x$), and use simple partial fractions to get the answer that you may then simplify using the log laws.

Problem Set 3

1. Write $y = 2x + 3$ make x the subject and substitute to find $f(y)$.

6. First work with $f(f(0))$ and $f(f(1))$.

Problem Set 4

4. Work with $x = \sec y$ and do not assume that $\tan y > 0$ but rather look at both possibilities.

9. Use similar triangles to relate the height of the cylinder to the dimensions of the cone.

10. Apply the Fundamental Theorem of Calculus, which says that $\frac{d}{dx}(\int_a^x f(t)\,dt) = f(x)$.

Problem Set 5

4. Integrating by parts twice will yield an expression for $2I$.

10. After integrating with respect to the new variable, use a suitable triangle to re-write the answer in terms of the original variable of integration.

Problem Set 6

4. Use Questions 1 and 2.

5. Put $y = x$ in suitable identities.

8. Make x the subject and then put $u = e^y$ to get a quadratic to solve in x.

Problem Set 7

4. Put $x = a \sinh t$.

7. Substitute the exponential series in the definiton of $\sinh x$.

8. Work with the equations $e^{\pm ix} = \cos x \pm i \sin x$; adding and subtracting to get expressions for $\cos x$ and $\sin x$ respectively. Then replace x by ix to get expressions for $\cosh x$ and for $\sinh x$.

Problem Set 8

2. Invert before taking derivatives.

3. take logs first.

5. Make use of the Inverse square law.

6. Use similar triangles to write the volume in terms of the height and then apply the Chain rule.

7. Fudamental theorem of calculus and the Chain rule.

8. Two standard limits arise here: can use L'Hopital's rule if you don't know them.

10. Can use $(\log_a b)(\log_b a) = 1$.

Problem Set 9

7. Use the formula $V = \pi \int_a^b (R^2 - r^2)\, dy$ where R and r are respectively the outer and inner radii of thin washers, which correspond to the $x-$co-ordinate of the corresponding boundary point of the rotating region, written in terms of the variable of integration, which is y.

Problem Set 10

1. Integrating by parts twices allows a formula for $2I$.

3. What is the integral of an odd function with limits of $\pm a$?

5. Since the integrand is not defined at the upper limit, we need to integrate between 0 and l and take the limit of this as $l \to 1^-$.

10. Primitives (anti-derivatives) can differ by a fixed constant.

Answers to the Problems

Problem Set 1

1. $4x\sec^2(x^2)\tan(x^2)$. 2. $\frac{x^2(7x-14)^{\frac{1}{3}}}{(1+x^2)^4}\left(\frac{2}{x}+\frac{1}{3x-6}-\frac{8x}{1+x^2}\right)$. 3. $6x+y-9=0$. 4. $\frac{2x}{10y+\cos y}$. 5. $-\frac{3yx^2+1}{28y^3+x^3}$. 6. $6cosec(2\cot 3x)\cot(2\cot 3x)\cdot cosec^2 3x$. 7. $-\frac{1}{\sqrt{1-x^2}}-\frac{x}{(1-x^2)^{\frac{3}{2}}}-\frac{\ln 2}{2^x}$. 8. $\sec x$. 9. e^{-1} . 10. $ad=bc$.

Problem Set 2

1. $x\sin x+\cos x+c$. 2. $\frac{1}{2}\ln 2$. 3. $-x+2\ln|1+x|+c$. 4. $\frac{1}{4}\sin 2x-\frac{1}{24}\sin 12x+c$. 5. $\tan\theta+\frac{2}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta+c$. 6. $\frac{184}{105}$. 7. $\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|+c$. 8. $x-\ln(1+e^x)+c$. 9. $x\ln x-x+c$. 10. 1.

Problem Set 3

1. $f(x)=\frac{1}{4}x^2-\frac{3}{2}x+\frac{13}{4}$. 2. $f(x)=\frac{x^2+1}{2x}$. 3. $e(x)=\cosh x$, $o(x)=\sinh x$. . 4. $f(x)=\pm\sqrt{2}x+\frac{1}{1\pm\sqrt{2}}$. 5. $f(x)=x\,\&\,f(x)=b-x$ 6. 1. 7. $y=\frac{-dx+b}{cx-a}$. 9. $\{x\in\mathbb{R}:x<2,x\neq\frac{3\pi}{4}+n\pi,n\in\mathbb{Z}\}$.10. $\emptyset$.

Problem Set 4

1. $-2\ln(a)\cdot a^{-2x}$. 2. nx^{n-1}. 3. $-\frac{2x}{x^4+2x^2+2}$. 4. $\frac{1}{|x|\sqrt{x^2-1}}$. 5. $(\pm 1,\frac{1}{\sqrt{2\pi e}})$. 6. $y=\frac{\sqrt{3}}{6}x+(2-\frac{\sqrt{3}\pi}{36})$. 7. $(12t^2+1)(\sec^2(4t^3+t))$. 8. $\sqrt{2}-4$. 9. $4:9$. 10. $\sin(x^2)$.

Problem Set 5

1. $\tan\frac{x}{2}+c$. 2. $\ln(\ln x|$. 3. $\frac{1}{3}\tan^{-1}\left(\frac{x+3}{3}\right)+c$. 4. $\frac{e^x}{2}(\sin x+\cos x)$. 5. $\frac{a^2-1}{2a^2\ln a}$. 6. $x\sin^{-1}x+\sqrt{1-x^2}$. 7. $\frac{\sin^5 x}{5}-\frac{2\sin^7 x}{7}+\frac{\sin^9 x}{9}+c$. 8. $\frac{1}{2}\left(x-\frac{\sin 4c}{4}\right)+c$. 9. π. 10. $-\frac{\sqrt{4-x^2}}{4x}+c$.

Problem Set 6

1. $\sinh x$ and $\cosh x$. 4. $\sinh^{-1}\left(\frac{x}{a}\right)$. 5. $\cosh^{-1}\left(\frac{x}{a}\right)$. 6. $\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)$. 7. $\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$. 8. $\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$.

Problem Set 7

1. $\sinh x$, $\cosh x$. 4. $\sinh^{-1}\left(\frac{x}{a}\right)+c$. 5. $\cosh^{-1}\left(\frac{x}{a}\right)+c$. 6. $\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)+c$. 7. $\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$.

Problem Set 8

1. $-\mathrm{cosec}^2(\sin x)\cos x$.2. $-\frac{1}{1+x^2}$. 3. $x^x(1+\ln x)$. 4. $5000\mathrm{m}^2$. 5. $d\left(1+\left(\frac{b}{a}\right)^{\frac{1}{3}}\right)^{-1}$. 6. $-\frac{25}{768\pi}\approx 1{\cdot}036\mathrm{m/min}$. 7. $\frac{\sin x}{2\sqrt{x}}$. 8.$\cos x$. 9. $y=0$ or $y=8(x-2)=8x-16$.10. $-\frac{1}{(\ln a)x(\log_a x)^2}$.

Problem Set 9

1. $\frac{1-t^2}{1+t^2}$.2. $\frac{2t}{1+t^2}$. 3. $\frac{2\,dt}{1+t^2}$. 4. θ $\ln|\tan\frac{\theta}{2}|$.5. $\ln\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}+c$. 6. $\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x-1}$,. $\ln\left|\frac{x}{x-1}\right|-\frac{1}{x}$. 7. $\frac{\pi}{6}$. 8. $\tan x-x$. 9. 0. 10. $x\arctan x-\frac{1}{2}\ln(1+x^2)$.

Problem Set 10

1. $-e^{-x}(x^2+2x+2)+c$. 2. $c-\frac{\sin x+\cos x}{2e^x}$. 3. 0. 4. $\int\left(1-\frac{1}{1+x^2}\right)dx = x-\arctan x+c$. 5. 2. 6. $\frac{\sqrt{2}}{2}\ln|\sec(x-\frac{\pi}{4})+\tan(x-\frac{\pi}{4})|+c$. 7. $(\cos 1)\ln|\sec(x+1)|-(\sin 1)x+c$. 8. $\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+c$. 9. $-\frac{a^2}{2}\arccos\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+c$.