Mathematics 204 Complex Variables

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This module builds on your existing knowledge of complex numbers to begin the study of functions of a complex variable, which holds many surprises. The first two problem sets continue the theme with the introduction of *stereographic projection* and problems involving the standard tool of using the complex exponentiation function and moving to real and imaginary parts. A complex variable can be viewed as a single variable and so the definition of differentiability of a real function extends to a complex one. However, at the same time it partakes of the nature of two variable functions in that the limit must exist through all directions of approach and the result is that *complex differetiablity* is very demanding in that the real and imaginary parts of the function must be linked through the *Cauchy-Riemann equations*, which are the subject of Set 3.

Contour integration, which the student will have seen in the context of vector functions, is introduced in Set 4 but the special nature of integration in the complex plane is explored through the *Cauchy integration formula* of set 5.

Since all differentiable functions of a complex variable are analytic and so can be represented by series, the topic of series arise often in the problem sets, including Set 6, where the *complex logarithm* function is also introduced. In Set 7 we study functions that are not analytic but can be represented by series that allows for negative powers of the complex variable z, the so-called *Laurent series*. In Set 8 the emphasis is on the *Cauchy Residue theorem* and its application in calculating integrals, including sometimes results for integrals along the real line. In Set 9 there is a variety of further problems making use of the techniques that have been introduced while Set 10 introduces the celebrated *Riemann zeta function* and some of its remarkable properties are to be found there.

Problem Set 1 Geometry of the complex numbers

1. Prove that

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

2. Show that the equation

$$az - \bar{a}\bar{z} + b = 0$$

where $a \in \mathbb{C} \setminus \{0\}$ and $b = ih$, $h \in \mathbb{R}$ is the cartesian equation of a straight line: $Ax + By + C = 0$, $A, B, C \in \mathbb{R}$.

3. Write the function

$$f(z) = 2x(1-y) + i(x^2 - y^2 + 2y)$$

as a function of $z = x + iy$.

4. Determine the set of points for which $\text{Re}((1+i)z) > 0$.

Stereographic projection Let the complex plane be represented by xy-cordinates and let S be the sphere $x^2 + y^2 + z^2 = 1$. Let the *north pole* be $N = (0, 0, 1)$. The stereographic projection W of $w = (x, y, 0) \in \mathbb{C}$ is the point where the line between N and w meets S. As $|w| \to \infty$, $W \to N$.

5. Show the line L through w and N is

$$L = \{(1-t)x, (1-t)y, t : -\infty < t < \infty\}$$

where $t = \frac{|w|^2 - 1}{|w|^2 + 1}$.

6. Find the coordinates of the point W where L meets S.

7. Given $W = (x_1, x_2, x_3)$, show that

$$w = \frac{x_1 + ix_2}{1 - x_3}.$$

8. Show that the square of the distance $d^2(W, W')$ between two points $W = (x_1, x_2, x_3)$ and $W' = (x_1', x_2', x_3')$ is given by

$$2 - 2(x_1x_1' + x_2x_2' + x_3x_3').$$

9. Show that

$$d(W, W') = \frac{2|w - w'|}{[(1 + |w|^2)(1 + |w'|)^2]^{\frac{1}{2}}}.$$

10. And similarly show that for $d(W, \infty)$, by which we mean $d(W, N)$, we have

$$d(W, \infty) = \frac{2}{(1 + |w|^2)^{\frac{1}{2}}}.$$

Problem Set 2 Further problems with polar and cartesian forms

1. Find the lowest value of n such that $z^n - 1$ has a root at $e^{\frac{i\pi}{10}}$.

2. Solve the equation

$$\cos z = \frac{1}{\sqrt{2}}.$$

3. Find the family of curves defined by the equation:

$$\mathbb{R}(\frac{1}{z}) = c,\ c \in \mathbb{R}.$$

4. Show that

$$\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cdots + \cos\frac{2(n-1)\pi}{n} = -1.$$

5. Show that

$$\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \cdots + \sin\frac{2(n-1)\pi}{n} = 0.$$

6. Find all the roots of the equation $z^n = (1+z)^n$, thereby showing that they all line on a single line in the complex plane.

7. Show that for any $z, w \in \mathbb{C}$ we have

$$\frac{z\overline{w} + \overline{z}w}{2} \leq |z||w|.$$

8. Deduce the triangle inequality from the result of Question 7.

9. For given complex numbers z_1, z_2 and z_3, prove that the relation

$$|z_1 - z_3|^2 = |z_1 - z_2|^2 + |z_2 - z_3|^2$$

is equivalent to

$$z_3 - z_2 = i\beta(z_2 - z_1),\ \text{for some } \beta \neq 0.$$

10. Find the images in the (u, v)-plane of lines parallel to the real axis and parallel to the imaginary axis under the mapping $z \mapsto \sin z$.

Problem Set 3 Cauchy-Riemann equations

The Cauchy-Riemann equations for a pair of functions $u(x,y)$, $v(x,y)$ are given by:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial v}{\partial y}.$$

The function of a complex variable $z = x + iy$ given by $f(z) = u(x,y) + iv(x,y)$ is differentiable (and then is smooth and has a power series) if and only if u and v satisfy the Cauchy-Riemann equations in which case

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

For Question 1-4, write the function $f(z)$ in the form $f = u + iv$, show that u and v satisfy the Cauchy-Riemann equations and find the derivative of $f'(z)$ in terms of z.

1. $f(z) = z^2$. 2. $f(z) = e^z$. 3. $f(z) = z^{-1}$.

4. The *principal value of the logarithm*: $\text{Log}(z) = \ln|z| + i\text{Arg}(z)$. Show differentiability for all z with positive real part.

5. Show that $f(z) = \overline{z}$ and $g(z) = |z|$ are not differentiable by showing that the corresponding Cauchy-Riemann equations do not hold.

6. If $f(z) = u + iv$ is differentiable, show that u is a *harmonic function,* which is to say satisfies *Laplace's Equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial v^2} = 0.$$

Find a *harmonic conjugate* of the (harmonic) function for the functions in Questions 7 and 8, which means the function $v(x,y)$ such that $u(x,y) + iv(x,y)$ is differentiable.

7.

$$u(x,y) = 2x(1-y).$$

8.

$$u(x,y) = 2y^3 - 6x^2y + 4x^2 - 7xy - 4y^2 + 3x + 4y - 4.$$

9. And again for $u(x,y) = \sinh x \sin y$, expressing $f(z) = u + iv$ as a function of z alone.

10. For $z = re^{i\theta}$ show that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Problem Set 4 Contour Integration

For the unit circle C defined by $|z| = 1$ find

1.
$$\int_C |z|\, dz.$$

2.
$$\int_C \mathrm{Re}(z)\, dz$$

3.
$$\int_C \overline{z}\, dz.$$

4. For $n \neq -1$ find
$$\int_C (z - z_0)^n\, dz$$
anti-clockwise around the circle $|z - z_0| = r$.

5. Repeat Question 4 with $n = -1$.

6. Find
$$\int_C |z|\, dz$$
where C is the quarter of the unit circle in the first quadrant traced anti-clockwise.

7. Repeat Question 6 with C now the connected line segments, the first of which goes from $(1, 0)$ to the origin while the second goes from $(0, 0)$ to $(0, 1)$.

8.
$$\int_\gamma z \sin z\, dz$$
where $\gamma = \{z \in \mathbb{C} : z = \frac{\pi}{2} e^{it},\ \mathrm{Im}(z) > 0\}(-\pi < t < \pi)$.

9. Show that
$$\int_{|z|=2} \frac{3z + 5}{z^2 + z}\, dz = 6\pi i.$$

10. Find
$$\int_i^{3i} z \sinh(z^2)\, dz.$$

Problem Set 5 Cauchy's Integral Formula

Cauchy's Integral formula If $f(z)$ is analytic inside a simply connected region D containing the simple closed path C and with $z_0 \in D$ then, traversing C anti-clockwise,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{z - z_0}.$$

Taking C to be a contour in D that encloses z_0, the value of the nth derivative is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}.$$

1. Use the formula to find $\int_C \frac{\cos z}{z}\,dz$ where C is the unit circle.

2. Prove the derivative formula by induction on n from the Cauchy integral formula, assuming we may differentiate with respect to z_0 inside the integral.

3. Integrate in the anti-clockwise sense around the circle $|z + i| = 1$:

$$\frac{z^2}{z^2 + 1}.$$

4. Repeat Question 3 for the circle $|z - i| = \frac{1}{2}$.

For Question 5 - 7 integrate around the unit circle in the positive sense.

5. $\frac{z^2}{(2z-1)^2}$ 6. $\frac{\cos z}{z^2}$. 7. $\frac{e^{z^3}}{z^3}$.

8. By substituting $w = z + 1$, show that

$$f(z) = \frac{4 - 6z}{2z^2 - 3z + 1} = \frac{5 - 3w}{(w - \frac{3}{2})(w - 2)}.$$

9. Use partial fractions on the previous expression to derive the Taylor series for $f(z)$ in Question 8.

10.

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n,$$

where B_n is called the nth *Bernoulli number*. Express z as a power series, written explicitly up to the term in z^5, in terms of Bernoulli numbers. Hence find the Bernoulli numbers up to and including B_4 .

Problem Set 6 Series problems and the logarithm function

1. Show by the Ratio test that the series

$$\sum_{n=0}^{\infty} (\frac{1+i}{2})^n$$

is convergent.

Find the centre and radius of convergence of the following power series.

2.

$$\sum_{n=0}^{\infty} 3^{2n}(1+z)^{3n}.$$

3.

$$\sum_{n=0}^{\infty} (n+1)(\frac{z}{5})^{n+1}.$$

4. Is this series convergent?

$$\sum_{n=1}^{\infty} \frac{(-1)^n n!}{e^n(1+i)^n}.$$

For $z \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ we define $\log z = \ln|z| + i \arg z$ and the *principal value of the logarithm* is $\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z)$.

5. Show that $e^{\mathrm{Log}(z)} = z$.

6. Find the values of Log at each of $i, -i, 1+i$ and $4i$.

7. Show that Log is not continuous on the negative real axis as follows. Take $\alpha > 0$ and consider the sequences $\{a_n = \alpha e^{i(\pi - \frac{1}{n})}\}$ and $\{b_n = \alpha e^{i(\pi + \frac{1}{n})}\}$. Show that both sequences approach α but that $\mathrm{Log}(a_n)$ and $\mathrm{Log}(b_n)$ do not approach the same value.

8. Show that Log is analytic on $\mathbb{C}^* \setminus \mathbb{R}^-$ by writing $z = re^{i\theta}$ and work in polar form with $\mathrm{Log}(z) = u(r,\theta) + iv(r,\theta)$.

9. The usual identity $\mathrm{Log}(z_1 z_2) = \mathrm{Log}(z_1) + \mathrm{Log}(z_2)$ does not generally hold (as a difference of 2π can arise between the two sides). However, show that for the multi-valued function $\log z$ we do have:

$$\log(z_1 z_2) = \log z_1 + \log z_2, \ \log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2.$$

10. Verify the first formula in detail when $z_1 = -2i$ and $z_2 = -i$ but show the formula fails if we replace log by Log in this case.

Problem Set 7 Laurent series

Laurent series If $f(z)$ is analytic on two concentric circles C_1 and C_2 and on the annulus between them then $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ where

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(w)\,dw}{(w-z_0)^{n+1}}\ (n \geq 0),\ a_n = \frac{1}{2\pi i}\oint_C (w-z_0)^{n-1} f(w)\,dw\ (n \leq -1)$$

where C is any simple closed curve that lies in the annulus and encloses z_0. (For $n \leq -1$ we shall write b_n in place of a_n for convenience.) The coefficient b_1 is the *residue of* $f(z)$ *at* z_0. If z_0 is a *pole of order* n (meaning that n is the largest subscript k such that $b_k \neq 0$) then

$$b_1 = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^n}\left((z-z_0)^n f(z)\right)|_{z=z_0}.$$

Alternatively, if $f(z) = \frac{P(z)}{Q(z)}$ where Q has a simple zero at z_0 (so that $(z-z_0)f(z)$ is analytic) then $b_1 = \frac{P(z_0)}{Q'(z_0)}$. If $f(z)$ has a *zero of order* n at z_0 then $1/f(z)$ has a pole of order n at z_0.

Find the Laurent series of the following functions.
1. $f(z) = \frac{1}{z-z^2}$, $z_0 = 0$.
2. $f(z) = \frac{3-z}{z^2-z^4}$, $z_0 = 0$.
3. $f(z) = \frac{1}{1-z^2}$, $z_0 = 1$.
4. Express

$$f(z) = \frac{z^2 - z + 3}{5(z+1)(z^2+4)}$$

in partial fractions, and find where $f(z)$ is not defined.
5. Hence find a Taylor series for the function $f(z)$ of Question 4 that is valid for $|z| < 1$.
6. For $1 < |z| < 2$ find a Laurent series for $\frac{1}{z+1}$.
7. Hence find a Laurent series for $f(z)$ in the region $1 < |z| < 2$.

Find the residues at the singular points of each of the following.
8.

$$\frac{\cos 2z}{z^4}$$

9.

$$\frac{z^2 + 11z + 1}{(z+1)^2(z-2)}$$

10. $\sec z$.

Problem Set 8 Cauchy's Residue Theorem

Cauchy's residue theorem: if $f(z)$ is analytic inside the simple closed curve C except for finitely many *poles* $z_1, z_2, \cdots, z_n$ inside C then:

$$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} Res_{(z\,=\,z_j)} f(z)$$

Identify the zeros z_0 of the following functions and the order of each such z_0 (the least n for which $f^{(n)}(z_0) \neq 0$).

1. $f(z) = z^3 - 2iz^2 - z$.

2. $f(z) = \tan z$.

With C the unit circle (traversed anti-clockwise) find:

3. $\int_C \frac{z}{4z^2-1}\,dz$ 4. $\int_C \frac{e^z}{\cos z}\,dz$.

Let Γ_R be the closed curve consisting of the circle centred at the origin of radius R, where $R > 1$, traversed anti-clockwise. Find

5. $\int_{\Gamma_R} \sin(z^2)\,dz$; 6. $\int_{\Gamma_R} \frac{e^{z^2}}{1-z}\,dz$ 7. $\int_{\Gamma_R} \frac{\sin(z-1)}{z^2-1}\,dz$.

8. If $|F(z)| \leq \frac{M}{R^k}$ for $z = Re^{i\theta}$, where $k > 1$ and M are constants, show that

$$\lim_{R\to\infty} \int_\Gamma F(z)\,dz = 0,$$

where Γ is the upper semicircle centred at the origin of radius R, traversed anti-clockwise.

9. Show that for $z = Re^{i\theta}$, $|f(z)| \leq \frac{M}{R^k}$, for some $M$and $k > 1$ and for all sufficiently large R, if $f(z) = \frac{1}{z^6+1}$.

10. Evaluate

$$\int_0^\infty \frac{dx}{1+x^6}.$$

Problem Set 9 Further problems

The *Wirtinger operators* are defined as

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}), \ \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}).$$

1. Show that for a differentiable function $f(z)$, $\frac{\partial f}{\partial \overline{z}} = 0$.

2. Show that $\frac{\partial}{\partial z}(z) = 1$ and $\frac{\partial}{\partial \overline{z}}(\overline{z}) = 1$.

Consider the integral:

$$\int_{|z|=1} \frac{dz}{(z-a)^n(z-b)^n}.$$

3. Evaluate the integral if $|a| < |b| < 1$;

4. Again but with $|a| < 1 < |b|$,

5. $1 < |a| < |b|$.

6. Evaluate

$$\int_{|z+1|=2} \frac{\sin z\, dz}{(z+2)^4}.$$

7. Find the Taylor series expansion around the origin of the function

$$f(z) = \cos\sqrt{z}.$$

8. Find

$$\int_\gamma \frac{e^{\pi z}}{(z^2+1)^2}\, dz$$

where γ is the curve with equation $4x^2 + y^2 - 2y = 0$.

Fundamental theorem of algebra Show that each non-constant polynomial $p(z) = a_0 + a_1 + \cdots + a_n z^n$ $(n \geq 1)$ has a complex root as follows. Suppose to the contrary that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Hence $f(z) = 1/p(z)$ is differentiable over all of $\mathbb{C}$.

9. Prove that $|f(z)|$ is bounded.

10. Show that if any bounded function $f(z)$ that is differentiable on all of $\mathbb{C}$ is constant by using the Cauchy integral formula to show that its derivative is 0. The contradiction represented by Questions 8 & 9 then shows that $p(z)$ has a root.

Problem Set 10 Riemann zeta function

The *Riemann zeta function* is defined as $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, where $s = x + iy$, $x > 1$.

1. Show that

$$\zeta(s) = \sum_{n=1}^{\infty} (\sum_{k=0}^{\infty} \frac{(\ln(n)s)^k}{k!})^{-1}.$$

Show *Euler' formula*

$$\zeta(s) = \Pi_p (1 - \frac{1}{p^s})^{-1}$$

where the product is taken over all primes as follows.

2. Write each term in the product as an infinite geometric series.

3. Multiply that series out and use the *Fundamental theorem of arithmetic* (uniqueness of prime factorization) to conclude the result.

4. Show that for $\mathbb{R}(s) > 0, s \neq 1$, we may extend ζ to the right half plane (except $s = 1$) by the following alternative formula, which agrees with the original definition for $\mathbb{R}(s) > 1$.

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum (-1)^{n+1} \frac{1}{n^s}.$$

Assume another identity of Euler that holds for $|z| < \pi$:

$$z\cot z = 1 - 2 \sum_{n=1}^{\infty} \frac{z^2}{n^2\pi^2 - z^2} \tag{1}$$

5. Show that (1) may be re-written as

$$z\cot z = 1 - 2 \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{n^{2k}} \right) \frac{z^{2k}}{\pi^{2k}} \tag{2}$$

6. Alternatively, show that

$$z\cot z = iz + \frac{2iz}{e^{2iz} - 1} \tag{3}$$

7. Use Question 10 of Set 5 to show from (3) that

$$z\cot z = 1 + \sum_{k=2}^{\infty} B_k \frac{(2iz)^k}{k!} \tag{4}$$

8. Deduce that

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}.$$

9. Use the relation for Bernoulli numbers:

$$\sum_{j=0}^{k} \binom{k+1}{j} B_j = 0, \; B_0 = 1$$

to find $\zeta(2)$, and $\zeta(4)$.

10. Given the functional equation:

$$\zeta(s) = 2(2\pi)^{s-1} \sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s), \; \mathbb{R}(s) < 1$$

show that for positive integer k, $\zeta(-2k) = 0$ and give an expression for $\zeta(-2k+1)$ in terms of the now known value of $\zeta(2k)$.

Hints for Problems

Problem Set 1

1. & 2. Use $|z^2| = z\overline{z}$ and properties of the modulus operator.
2. Substitute $z = x + iy$ and see where it leads.
3. If all else fails use $x = \frac{z+\overline{z}}{2}$ and $y = \frac{z-\overline{z}}{2i}$ and simplify.

Problem Set 2

4. & 5. First find the sum $\sum_{k-0}^{n-1} e^{\frac{2\pi i}{n}}$ and then take the real and imaginary parts respectively.

6. Solve for z and then use that $e^{\frac{ik\pi}{n}} - e^{-\frac{ik\pi}{n}} = 2i \sin k\pi$ to simplify the result.

7. Use that $z + \overline{z} = 2\mathbb{R}(z)$ and $\mathbb{R}(z)^2 \leq |z|$; substitute in the equality to gain the required result.

8. Begin from $(z+w)^2 = (z+w)(\overline{z+w})$.

9. Put $\mu = \frac{z_3 - z_2}{z_2 - z_1}$ and show that $z_3 - z_1 = (1+\mu)(z_2 - z_1)$. Substituting in the given equality now leads to the conclusion that μ is purely imaginary.

10. In each case represent $\sin z$ in the form $u + iv$ and find the equation relating u and v.

Problem Set 3

9. Express $\cosh z$ in the form $u + iv$.

10. Intergrate $\frac{\partial v}{\partial y}$ wrt y and equate to $\frac{\partial u}{\partial x}$; remember to include a function $\phi(x)$ as a result of the integration. Then determine $\phi(x)$ using the second CR equation.

Problem Set 4

1 - 3. Using the standard parametrization $z = e^{it}$ $(0 \leq t \leq 2\pi)$. Simplify $|z|$ on C.

2. $\text{Re}(z) = \cos t = \frac{1}{2}(e^{it} - e^{-it})$.

4. Use $z = z_0 + \rho e^{it}$ $(0 \leq t \leq 2\pi)$.

8. Find an anti-derivative and evaluate via the endpoints of the contour.

9. Use partial fractions and the *Principle of Deformation*, which says that if one contour is contained inside another and the function to be integrated is analytic between the contours then the integral of the function is the same around both contours.

10. Find an anti-derivative of the integrand and evaluate directly.

Problem Set 5

3. & 4. Factorize the denominator and work with the root inside the contour of integration when applying the Cauchy formula.

5. & 6. This time you will need the derivative formula with $n = 1$.

7. Again but with $n = 2$.

10. Cross-multiply and use the known exponential series to find the Bernoulli numbers by equating coefficients.

Problem Set 6

1-4 Remember the *ratio test*: the radius of convergence R satisfies $R^{-1} = \lim_{n\to} \left|\frac{z_{n+1}}{z_n}\right|$ (if this limit exists).

8. Use the results of Question 10 on Set 3.

Problem Set 7

1. & 2. Factorize as a simple rational function and a geometric series to find the Laurent series.

3. First substitute $w = z - z_0$so that the series is centred at $w = 0$. Write $\frac{1}{w+2}$ as $\frac{1}{2(1-(-w/2))}$ and expand as a geometric series.

4 - 7. After finding the points where $f(z)$ is not defined, you will find there are three cases to consider: (i) $|z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$. For (ii) use that $\frac{1}{z+1} = \frac{1}{z(1+\frac{1}{z})} = \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right)$, provided $|\frac{1}{z}| < 1 \Leftrightarrow |z| :> 1$.

8. Here we have a pole of order 4.

9. Here we have a pole of order 2 and a *simple pole* (pole of order 1) so in the latter case the Q/P formula may be applied.

10. Infinitely many poles but they are all simple.

Problem Set 8

3. With simple poles, the Q/P formula works well.

7. One of the two singularities is a *removable* singularity, meaning that the limit of the function as the singular point is approached does exist and so, by defining the function so as to be continuous there, it can be disregarded in the calculation.

10. Use as contour the semicircle in the upper half plane with diameter $[-R, R]$. Apply the Residue Theorem and Question 9.

Problem Set 9

3. Both singular points are within the contour: replace the contour by two small contours around each, apply the CIF to each one and add to get the answer.

4. Now only one singular point lies within γ.

6. Another application of the CIF.

7. This question is a bit of a cheat as in general $\sqrt{z}$ is double-valued and there is no 'positive' root. However, in this case substitute directly into the series for cosine and there is no ambiguity because powers involved are even.

8. You will need to find the exact nature of γ first in order to identify which singularities lie inside the contour.

10. Show that $|f'(z_0)|$ is arbitrary small by applying the CIF for the first derivative for the contour circle $|z - z_0| = r$ and then letting $r \to \infty$.

Problem Set 10

1. Use the exponential series.

4. Multiply out $(1 - 2^{1-s})\zeta(s) = (1 - 2 \cdot \frac{1}{2^s})(1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots)$.

5. Expand $\frac{z^2}{n^2\pi^2 - z^2} = \frac{z^2}{n^2\pi^2} \cdot \frac{1}{1 - \frac{z^2}{n^2\pi^2}}$.

6. $\cos z = \cosh(iz)$ and $\sin z = \frac{1}{i}\sinh(iz)$.

7. Use Question 9 of Set 5.

Answers to the Problems

Problem Set 1

2. $A = 2\mathrm{Im}(a)$, $B = 2\mathrm{Re}(a)$ and $C = h$. 3. $iz^2 + 2z$. 4. $y < x$. 5. 6. $\left(\frac{2x}{|w|^2+1}, \frac{2y}{|w|^2+1}, \frac{|w|^2-1}{|w|^2+1}\right)$. 7. $\frac{x_1}{1-x_3} + \frac{ix_2}{1-x_3} = \frac{x_1+ix_2}{1-x_3}$.

Problem Set 2

1 20. 2. $\{z = x + iy \in \mathbb{C} : x = 2n\pi \pm \frac{\pi}{4}, y = 0, n \in \mathbb{Z}\}$. 3. A circle centred at $(\frac{1}{2c}, 0)$ of radius $\frac{1}{2c}$. 6. $z = -\frac{1}{2}\left(1 + i\cot\frac{k\pi}{n}\right)$ 10. $\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1$; $\frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c} = 1$.

Problem Set 3

1. $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$. $f'(z) = 2z$. 2.$u(x, y) = e^x \cos y$, $v(x, y) = e^x \sin y$. $f'(z) = e^z$. 3. ,$f'(z) = -\frac{1}{z^2}$.4. $u(x, y) = \frac{1}{2}\ln(x^2 + y^2)$, $v(x, y) = \arctan\left(\frac{y}{x}\right)$ $f'(z) = z^{-1}$. 7.$v(x, y) = x^2 - y^2 + 2y + c$. 8.$-6xy^2 + 8xy + \frac{7}{2}y^2 + 3y + 2x^3 + \frac{7}{2}x^2 + 4x + c$ $(c \in \mathbb{R})$. 9. $\sinh x \sin y - i\cosh x \cos y + ic$, $(c \in \mathbb{R}) = -\cosh z$.

Problem Set 4

1. 0. 2. πi. 3. $2\pi i$. 4. 0. 5. $2\pi i$. 6. $i - 1$. 7. $\frac{i-1}{2}$. 8. 2. 9. $6\pi i$. 10. $\frac{\cosh 9 - \cosh 1}{2}$.

Problem Set 5

1. $2\pi i$. 2. π. 3. $-\pi$. 4. $\frac{\pi i}{2}$. 5. 0 . 6. 0. 7. $\frac{\frac{2}{3}}{1-\frac{2}{3}w} + \frac{1}{1-\frac{w}{2}}$. 8. $\sum_{n=0}^{\infty}((\frac{2}{3})^{n+1} + (\frac{1}{2})^n)(z + 1)^n$; $|z + 1| < \frac{3}{2}$. 9. & 10. $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$; $B_3 = 0$; $B_4 = -\frac{1}{30}$.

Problem Set 6

2. Centre at $z = -1$, radius of convergence $\frac{1}{\sqrt[3]{3}}$. 3. Centre at 0, radius of convergence is 5. 4. No. 6. $\frac{i\pi}{2}$, $\frac{-i\pi}{2}$, $\frac{1}{2}\ln 2 + i\frac{\pi}{4}$, $2\ln 2 + i\frac{\pi}{2}$.

Problem Set 7

1.$\sum_{n=-1}^{\infty} z^n$, $|z| < 1$. 2. $= \frac{3}{z^2} - \frac{1}{z} + 3 - z + 3z^2 - z^3 + 3z^4 - z^5 + 3x^6 - z^7 + \cdots$, so centre is 0, radius of convergence 1.3. $-\frac{1}{2(z-1)} + \frac{1}{4} - \frac{z-1}{8} + \frac{(z-1^2}{16} - \cdots + (-1)^n \frac{(z-1)^n}{2^{n+2}} + \cdots$, $|z-1| < 2$. 4. $\frac{1}{5}\left(\frac{1}{z+1} - \frac{1}{z^2+4}\right)$ and $f(z)$ is not defined at $z = -1$ and $z = \pm 2i$. 5. $\sum_{n=0}^{\infty}(-1)^n z^n$, when $|z| < 1$. 6. $\frac{1}{z+1} = \frac{1}{z(1+\frac{1}{z})} = \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right)$. 7. $\frac{1}{5z} - \frac{2}{5z^2} + \frac{1}{5z^3} + \frac{3}{5z^4} + \frac{1}{5z^5} - \frac{17}{5z^6} + \cdots$. 8. 0. 9. -2. 10. ± 1 according as $z = 2n\pi + \frac{\pi}{2}$ or $z = (2n+1)\pi + \frac{\pi}{2}$.

Problem Set 8

1. simple pole at 0, pole of order 2 and i. 2. simple poles at $n\pi$ $(n \in \mathbb{Z})$. 3. $\frac{\pi i}{2}$. 4. 0. 5. 0. 6. $-e$. 7. $\frac{1}{2}\sin 2$. 10. $\frac{2\pi}{3}$.

Problem Set 9

3. 0. 4. $\frac{2\pi i}{(n-1)!}(-1)^{n-1}\frac{(2n-2)!}{(n-1)!} \cdot \frac{1}{(a-b)^{2n-1}}$. 5. 0. 6. $-\frac{\pi i}{3}\cos 2$. 7. $= \sum_{n=0}^{\infty}(-1)^n \frac{z^n}{(2n)!}$. 8. $\frac{\pi}{2}e^{\pi i}(1 - \pi i)$.

Problem Set 10

9. $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$. 10 $\zeta(-2k) = 0$, $\zeta(1-2k) = \frac{(-1)^{k+1}\Gamma(2k)B_{2k}}{4\pi^2}$.