Mathematics 106 Matrices & Linear Algebra

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The British QAA (Quality Assurance Agency) lists calculus and linear algebra as core to any Mathematics degree. Students entering university level mathematics normally have learnt a fair bit of calculus. However matrix methods and some abstract linear algebra are also staples of a First Year mathematics course and these topics may be quite new. Matrix algebra underlies nearly all large scale calculation in the sciences, including social sciences.

There are two main aspects to this module. The first is matrix calculations in all their forms. (It is *matrix multiplication* that is the big new idea.) The student will need to be familiar with solving systems of linear equations and how to carry this out in matrix notation. This involves the method of *Gaussian elimination*, which is the technique of elimination of variables applied in matrix form. The advantage of the matrices is that they allow the bulk of the elimination to take place just with simple arithmetic operations on the coefficients, without intrusion of any algebraic symbols, by reducing the coefficient matrix to *echelon form*. We shall also assume that the student is familiar with the notion of the *determinant* $|A|$ of a square matrix A, at least for matrices of size up to 3×3 and that $|AB| = |A| \cdot |B|$. The student will also be asked to invert square matrices, which can be done through determinant methods and also through elimination techniques (the latter being generally preferred). *Eigenvalues* and *eigenvectors* of a matrix represent another fundamental pair of ideas that you will meet.

The second aspect of linear algebra in the module is that of *vector spaces* and *linear mappings* between them. In this exerecise set, the underlying spaces can all be taken to lie in $\mathbb{R}^n$. However, although we do not call upon them, the student should be familiar with the *axioms of a vector space* and the associated definitions as there are many exercises based on the notion of *independent set*, which lead to the related notions of *spanning set*, and *dimension of a vector space*. We do tie these abstract ideas to corresponding notions in matrix algebra through the exercises. In particular, the fundamental fact known as the *Steinitz Exchange Lemma* is assumed as background knowledge in some problems.

Vector spaces are often the first abstract algebraic environment that a university student is asked to work within. Argument from axioms to theorems is not much dealt with at school level and can take a real effort to get used to. The advantage of vectors spaces as a vehicle for this training is that many of the important results are quite intuitive once a few basic ideas are grasped and the proofs are themselves fairly simple.

Problem Set 1 Matrices and determinants

1. For the matrices

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & -2 \\ 0 & 3 \\ -1 & 0 \end{bmatrix}$$

find, if they exist, the products AB and BA.

2. Find the determinants of the matrix products in Question 1.

3. Let $\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}$ denote the rows of BA ordered from top to bottom. Find constants a and b such that $\mathbf{r_3} = a\mathbf{r_1} + b\mathbf{r_2}$.

4. Evaluate the following determinant by the first row expansion and by second column expansion.

$$\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & \frac{1}{2} \\ -1 & 4 & 0 \end{vmatrix}.$$

5. Find matrices A and B such that AB is the identity matrix yet $B \neq A^{-1}$.

6. Find a matrix all of whose entries are non-zero that squares to the zero matrix.

7. Find the inverse of the matrix

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

8. Find the inverse of the matrix:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

9. Find the rank of

$$A = \begin{bmatrix} -1 & -2 & 2 \\ 2 & 1 & 4 \\ 0 & 1 & -1 \\ -3 & 2 & 3 \end{bmatrix}.$$

10. Evaluate the complex determinant:

$$\begin{vmatrix} a+bi & c+id \\ id-c & a-ib \end{vmatrix}$$

and hence deduce that if integers m and n are each the sum of four squares then so is mn.

Problem Set 2 Systems of linear equations

Solve the following systems of equations using Gaussian elimination (the row reduction method).

1.

$$\begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & + & x_3 & - & x_4 & & & = & 2 \\ 2x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & & & = & 5 \\ x_1 & - & x_2 & + & 2x_3 & + & 3x_4 & + & x_5 & = & 4 \end{array}$$

2.

$$\begin{array}{rrcrcrcr} & & & & x_3 & + & x_4 & = & 0 \\ -2x_1 & -4x_2 & + & x_3 & & & & = & -3 \\ 3x_1 & +6x_2 & - & x_3 & + & x_4 & & = & 5 \end{array}$$

3. Use row reduction to find the inverse of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 3 & 6 & 10 \end{bmatrix}.$$

4. Use *Cramer's rule* to solve the following system of equations:

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 6 \\ 3x & + & 4y & + & 5z & = & 22 \\ 3x & + & 6y & + & 10z & = & 31 \end{array}$$

5. Use Cramer's rule to find the inverse of the matrix A of Question 3.

6. Show that a square *integer matrix* (one whose entries are integers) A has an integer matrix inverse if and only if $|A| = \pm 1$.

7. Show from Cramer's rule that A^{-1} exists if and only if $|A| = 0$.

8. Show that the solution set to $A\mathbf{x} = \mathbf{b}$ has the form $\mathbf{a} + S$ where $\mathbf{a}$ is any solution to the given equation and S is the set of solutions of the corresponding *homogenous system* $A\mathbf{x} = \mathbf{0}$.

9. Use Question 8 to show that the system of equations $A\mathbf{x} = \mathbf{b}$ has either no solution, a unique solution (vector) or infinitely many solutions.

10. Show that $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if $|A| \neq 0$.

Problem Set 3 Eigenvalues and Eigenvectors

The *eigenvalues* λ of a square matrix A are the solutions to $|A-\lambda I| = 0$ and the corresponding *eigenvectors* are the non-zero solutions $\mathbf{x}$ to $A\mathbf{x} = \lambda\mathbf{x}$. If all these λ are real and distinct then A may be *diagonalized* as $A = PDP^{-1}$ where D is the diagonal matrix of eigenvalues and the colums of P are eigenvectors of A listed in the order corresponding to the order of eigenvalues.

1. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} -17 & 30 \\ -10 & 18 \end{bmatrix}.$$

2. Find an eigenvector corresponding to each eigenvalue of Question 1.

3. Hence diagonalize A in the form $A = PDP^{-1}$.

4. Use your answer to Question 3 to calculate A^6.

5. Show that two similar matrices share the same eigenvalues.

6. Show that if one of the square matrices A and B is non-singular then AB and BA are similar but this is not true in general.

7. The *trace* of a square matrix A is the sum of the entries on its principal diagonal. Show for square matrices A and B that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

8. Use Question 7 to prove that similar matrices share the same trace.

9. Let $A = [a_{ij}]$ be an $n \times n$ *upper triangular matrix*, meaning that $a_{ij} = 0$ for all $i < j$. Show that

$$|A| = \Pi_{k=1}^{n} a_{kk}.$$

10. Show that the eigenvalues of the matrix of Question 9 are precisely the diagonal entries a_{kk}.

Problem Set 4 Eigenvalues and eigenvectors applications

Our problem is to find out the number of integers with n digits that can be formed using members of the set $\{2, 3, 7, 9\}$ which are divisible by 3. Let x_n ($n \geq 1$) be the required number and let y_n and z_n be the number of integers of length n formed from our set that are conguent to 1 and to 2 respectively modulo 3. Write $\mathbf{x_n}$ for the column vector $(x_n, y_n, z_n)^T$.

1. Write down the values of $\mathbf{x}_0$ and of $\mathbf{x}_1$.

2. Write down recurrence relations for x_n, y_n and z_n respectively in terms of x_{n-1}, y_{n-1} and z_{n-1}.

3. Express the result of Question 2 as a single matrix relation $\mathbf{x}_n = A\mathbf{x}_{n-1}$ and hence deduce that $\mathbf{x}_n = A^n\mathbf{x}_0$.

4. Find the eigenvalues of the matrix A.

5. Find the eigenvectors of the matrix A.

6. Hence diagonalize A as $A = PDP^{-1}$ for a diagonal matrix D.

7. Now find the form of an arbitrary power A^n of A.

8. Finally, write down $\mathbf{x}_n$ as a function of n alone.

9. Prove the result of Question 8 as regards x_n by induction on n.

10. For a square matrix A define

$$e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}.$$

For

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ show that } e^{tA} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

Problem Set 5 Positive definite matrices and quadratic forms

An $n \times n$ symmetric matrix A with real entries is called *positive definite* if $\mathbf{x}^T A\mathbf{x} > 0$ for every non-zero vector $\mathbf{x} \in \mathbb{R}^n$. More generally, an $n \times n$ *Hermitian matrix* A (one that equals its own conjugate transpose) is positive definite if $z^* Az > 0$ for all non-zero column vectors z of n complex numbers, where z^*denotes the conjugate transpose of z.

1. By calculating the expression $\mathbf{x}^T I\mathbf{x}$ explicitly, show that the 2×2 identity matrix I is positive definite.
2. More generally, show that the $n \times n$ identity matrix is Hermitian and positive definite.
3. Show that the real symmetric matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

is positive definite.

4. Show that the following matrix A is not positive definite where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

5. Show that any eigenvalue λ of a positive definite matrix A is itself positive.

An n-ary *quadratic form* over the reals is an homogeneous polynomial of degree 2 in n variables:

$$q(x_1, x_2, \cdots, x_n) = \sum_{i,j=1}^{n} a_{ij}x_ix_j = \mathbf{x}^T A\mathbf{x}$$

where $A = (a_{ij})$ and $\mathbf{x}$ is the column vector of the variables x_i.

6. Represent the quadratic form $Q(x, y) = 5x^2 - 10xy + y^2$ in the form $\mathbf{x}^T A\mathbf{x}$.
7. Repeat Question 6 but with A chosen so that A is a symmetric matrix M.
8. Show generally how the coefficient matrix A of an n-ary quadratic form may be replaced by a symmetric matrix M so that $q(x_1, x_2, \cdots, x_n) = \mathbf{x}^T M\mathbf{x}$.
9. Let $\mathbf{v}_1$,$\mathbf{v}_2$ be two eigenvectors corresponding to eigenvalues λ_1, λ_2 of a symmetric matrix A. Show that $\lambda_1(\mathbf{v}_1 \bullet \mathbf{v}_2) = \lambda_2(\mathbf{v}_1 \bullet \mathbf{v}_2)$.
10. Under the conditions of Question 9, deduce that if $\lambda_1 \neq \lambda_2$ then the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are mutually orthogonal.

Problem Set 6 Matrices and Analytical Geometry

The matrices of a linear transformation have as their columns the images of the list members of the *standard basis vectors* (which are in 3d, the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$). As usual, function composition is carried out from right to left, so acting the matrix AB means first act B, then A.

1. Find the matrix that represent a rotation of 30° clockwise about the origin and write down its inverse.

2. Find the matrix that represents a reflection in the line through the origin that makes an angle $\frac{\pi}{8}$ with the x-axis and write down its inverse.

3. What is the matrix that transforms the plane by sliding each point to the left of the y-axis vertically upwards, and those to the right downwards, a distance equal to five times the distance of the point from the y-axis?

4. Find the matrix A for a rotation of θ $(0 \leq \theta \leq 2\pi)$ anti-clockwise about the origin and give the matrix for the composition of this transformation with another rotation about the origin through an angle ϕ.

5. Write down the matrix for reflection R in the line making an angle θ $(0 \leq \theta \leq \pi)$ with the x-axis.

6. Find the matrix that represents the rotation of Question 4 (with angle θ) followed by the reflection of Question 5 (with angle ϕ), in that order, and identify the composite transformation geometrically.

7. Use the result of Question 6 to find the values of $\cos\frac{5\pi}{12}$ and $\sin\frac{5\pi}{12}$.

8. Repeat Question 6 but reversing the order of the two transformations.

9. Find the matrix representing the composition of two reflections in lines through the origin with respective angles of θ and ϕ and interpret it geometrically.

10. Write down the matrix A that represents the linear transformation in which $\mathbf{i} \mapsto \mathbf{j}$, $\mathbf{j} \mapsto \mathbf{k}, \mathbf{k} \mapsto \mathbf{i}$, and write down its inverse, A^{-1} and find the eigenvalues and corresponding eigenvectors of A.

Problem Set 7 Linear Independence and Bases

1. Determine whether or not the set

$$\{(-1, 2, 0, -3), (-2, 1, 1, 2), (2, 4, -1, 3)\}$$

is an independent subset of $\mathbb{R}^4$.

2. Find a basis for the subspace of $\mathbb{R}^4$ generated by

$$A = \{(1, -1, 3, 2), (-1, 3, -2, 2), (2, 1, 2, -1), (-1, 0, 2, 7)\}.$$

3. Find a basis for the hyperplane $2x_1 - 3x_2 + 4x_3 - x_4 = 0$.

4. Suppose that $U = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\}$ is an independent subset of a vector space V. Show that any $\mathbf{u} \in \langle U \rangle$ has a unique set of *coordinates* $(a_1, a_2, \cdots, a_k)$ with respect to the basis U: 'coordinates' meaning that

$$\mathbf{u} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_k\mathbf{u}_k.$$

5. Let $A = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\}$ be a finite subset of a vector space V, listed in that specific order. Show that A is independent if and only if no member of A is a linear combination of the vectors that precede it in the list.

The *row space* (resp. *column space*) of an $m \times n$ matrix M with real entries is the subspace R of $\mathbb{R}^n$ (resp. C of $\mathbb{R}^m$) spanned by the rows (resp. columns) of M. The *row rank* (resp. *column rank*) of a matrix M is the maximum number of independent rows (resp. columns) of M. Corresponding sets of rows and columns are then respectively bases for the row space and column space of M.

6. Show that the row space of M is *invariant* (that is, does not change) under each of the three row operations of row interchange, multiplying a row by a non-zero constant a, and of adding a multiple of on row to another.

Recall that any matrix can be reduced via row operations to *echelon form* in which the height of each stair is 1 row, below the staircase all entries are 0, in each corner of each stair the entry is 1, and above and below any such *pivotal* 1 all entries are 0.

7. Show that the row rank of any $m \times n$ matrix M equals is $m - k$ where k is the number of zero rows in the echelon form of M.

8. Show that the set of pivotal columns of the echelon form E of the matrix M form an independent subset of $\mathbb{R}^m$.

9. Use Question 7 and 8 to show that the row rank of M is no more than its column rank.

10. Hence deduce that the row and column ranks of any matrix are equal.

Problem Set 8 Linear mappings

Thoughout let $L : U \to V$ denote a linear mapping between vector spaces, so that $L(a\mathbf{u} + b\mathbf{v}) = aL(\mathbf{u}) + bL(\mathbf{v})$.

1. Show that $L(\mathbf{0}) = \mathbf{0}$.

2. Show that for any k-fold linear combination:

$$L(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_k\mathbf{u}_k) = a_1L(\mathbf{u}_1) + a_2L(\mathbf{u}_2) + \cdots + a_kL(\mathbf{u}_k).$$

3. Show that the kernel of L, $\ker(L) = \{\mathbf{u} \in U : L(\mathbf{u}) = \mathbf{0}\}$ is a subspace of the domain space U.

4. Show that the range $L(U)$ is a subspace of V.

5. Show that L is one-to-one if and only if $\ker(L) = \{\mathbf{0}\}$.

6. Take $U = \mathbb{R}^n$ and $V = \mathbb{R}^m$. Let M be an $m \times n$ matrix and let L be defined by $L(\mathbf{u}) = M\mathbf{u}$. Show that the columns of M form a spanning set for the range space $L(U)$.

7. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ and let $\mathbf{c}_j = L(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth standard basis vector of $\mathbb{R}^n$. Let M be the $m \times n$ matrix whose jth column is $\mathbf{c}_j$. Show that $L(\mathbf{u}) = M(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{R}^n$.

8. Let U, W be subspaces of a vector space V. Show that $U \cap W$ is also a subspace of V.

9. Let $A, B \subseteq V$ with $A \subseteq \langle B \rangle$, the subspace of V generated by the set B. Show that $\langle A \rangle \subseteq \langle B \rangle$.

10. Show that any two bases B_1 and B_2 of a finite dimensional vector space V have the same number of elements.

Problem Set 9 Sums and direct sums of subspaces

Let V be a finite dimensional vector space throughout.

1. Let U, W be subspaces of V. Show that

$$U + W = \{\mathbf{u} + \mathbf{w} : \mathbf{u} \in U,\, \mathbf{v} \in V\}$$

is a subspace of V.

2. Let $L : U \to V$ be a linear mapping between finite dimensional vector spaces. Prove that

$$\dim(\text{kernel}(L)) + \dim(\text{range}(L)) = \dim(\text{domain}(L)).$$

3. Prove that any independent subset A of a finite dimensional vector space may be extended to a basis for V.

4. Consider the equation systems $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$. Show that the solution set of the latter is $U + \mathbf{v}$ where U is the subspace of solutions of the *homogeneous system* $A\mathbf{x} = \mathbf{0}$ and $\mathbf{v}$ is any solution of the *inhomogeneous* system $A\mathbf{x} = \mathbf{b}$.

5. Let U be a subspace of V. Show that $\dim(U) \leq \dim(V)$ with equality if and only if $U = V$.

6. Prove that any maximal independent subset of a spanning set for V is a basis for V.

7. Prove that for subspaces U, W of a finite dimensional vector space V:

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

8. Show that the representation $\mathbf{x} = \mathbf{u} + \mathbf{v}$ of a member $\mathbf{x} \in U + W$ is unique if and only if $U \cap W = \{\mathbf{0}\}$, in which case we call $U + W$ the *direct sum* of U and W, denoted by $U \oplus W$.

9. Let U be a subspace of $\mathbb{R}^n$. Define

$$U^{\perp} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{u} \bullet \mathbf{x} = 0 \,\forall\, \mathbf{u} \in U\}.$$

Show that $U^{\perp}$ is a subpsace of V and that $U \cap U^{\perp} = \{\mathbf{0}\}$.

10. Show that

$$U \oplus U^{\perp} = V.$$

Problem Set 10 Orthonormal bases

1. Let $A = \{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ be an ordered *orthonormal basis* (a basis of unit vectors that are pairwise mutually orthogonal, written in a fixed order) of a subpsace of $\mathbb{R}^n$ and let

$$\mathbf{v} = a_1\mathbf{v_1}{+}a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k.$$

Show that $a_i = \mathbf{v} \bullet \mathbf{v}_i$ $(1 \le i \le k)$.

2. Prove that an orthonormal set $A = \{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$ is independent.

3. Show that $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ is an ordered orthonormal basis for $\mathbb{R}^2$ where $\mathbf{v}_1 = \frac{1}{2}(\sqrt{3}, 1)$ and $\mathbf{v}_2{=}\frac{1}{2}(-1, \sqrt{3})$. Find the B -coordinates of $\mathbf{v} = (-2, 3)$.

Gram Schmidt algorithm Let $A = \{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ be a basis of a subspace V of $\mathbb{R}^n$. Define $\{\mathbf{w}_1, \cdots, \mathbf{w}_k\} \subseteq V$ by

$$\mathbf{w}_1 = \mathbf{v}_1,\ \mathbf{w}_i = \mathbf{v}_i - \frac{\mathbf{v}_i \bullet \mathbf{w}_1}{\mathbf{w}_1 \bullet \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{v}_i \bullet \mathbf{w}_2}{\mathbf{w}_2 \bullet \mathbf{w}_2}\mathbf{w}_2 - \cdots - \frac{\mathbf{v}_i \bullet \mathbf{w}_{i-1}}{\mathbf{w}_{i-1} \bullet \mathbf{w}_{i-1}}\mathbf{w}_{i-1},\ 2 \le i \le k.$$

4. Verify that the set $B = \{\frac{\mathbf{w}_1}{||\mathbf{w}_1||}, \cdots, \frac{\mathbf{w}_k}{||\mathbf{w}_k||}\}$ is an orthonormal basis for V.

5. Find an orthonormal basis for the solution space of the system of homogeneous equations:

$$\begin{array}{rcrcrcrcl} x & + & y & + & z & + & w & = & 0 \\ -x & + & y & & & + & w & = & 0. \end{array}$$

6. Find the change-of-basis matrix P from $\mathbf{B}$- to $\mathbf{C}$-corodinates for the bases of a certain subspace V of $\mathbb{R}^5$ where

$$\mathbf{B} = \{(-4, 5, -1, 0, -1)^T, (1, -3, 2, 2, 5)^T, (1, -2, 1, 1, 3)\},$$

$$\mathbf{C} = \{(1, 0, -1, 0, 4)^T, (0, 1, -1, 0, 3)^T, (0, 0, 0, 1, 5)^T\},$$

and hence find the $\mathbf{C}$-coordinates of the vector $\mathbf{v}$ the $\mathbf{B}$-coordinates of which are $(2, -1, 0)$.

7. Diagonalize the matrix A of the transformation whose matrix with respect to the standard basis is

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix},$$

and hence show that $A^n\mathbf{v}$ approaches the direction of the eigenvector corresp-sonding to *dominant eigenvalue* (the eigenvalue of greatest magnitude) for all vectors that are not multiples of the eigenvector of the smaller eigenvalue.

8. Let $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ denote an orthonormal basis for $\mathbb{R}^n$. Show that for any two members $\mathbf{v}, \mathbf{w}$ of $\mathbb{R}^n$

$$\mathbf{v} \bullet \mathbf{w} = (\mathbf{v} \bullet \mathbf{u}_1)(\mathbf{w} \bullet \mathbf{u}_1) + \cdots + (\mathbf{v} \bullet \mathbf{u}_n)(\mathbf{w} \bullet \mathbf{u}_n); \text{ and that}$$

$$\mathbf{v} \bullet \mathbf{v} = ||\mathbf{v}||^2 = (\mathbf{v} \bullet \mathbf{u}_1)^2 + \cdots + (\mathbf{v} \bullet \mathbf{u}_n)^2.$$

9. Use the *Parseval equality* in Question 8 to deduce *Bessel's inequality*:

$$(\mathbf{v} \bullet \mathbf{u}_1)^2 + \cdots + (\mathbf{v} \bullet \mathbf{u}_k)^2 \leq ||\mathbf{v}||^2.$$

10. The *least squares approximation* to a set $(x_1, y_1), \cdots, (x_n, y_n)$ in $\mathbb{R}^2$ is the line $y = mx+b$ where for $\mathbf{1} = (1, 1, \cdots 1)$, $\mathbf{x} = (x_1, \cdots, x_n)$, $\mathbf{y} = (y_1, \cdots, y_n)$

$$m = \frac{(\mathbf{1} \bullet \mathbf{1})(\mathbf{x} \bullet \mathbf{y}) - (\mathbf{1} \bullet \mathbf{x})(\mathbf{1} \bullet \mathbf{y})}{(\mathbf{1} \bullet \mathbf{1})(\mathbf{x} \bullet \mathbf{x}) - (\mathbf{1} \bullet \mathbf{x})^2} \quad b = \frac{(\mathbf{x} \bullet \mathbf{x})(\mathbf{1} \bullet \mathbf{y}) - (\mathbf{x} \bullet \mathbf{y})(\mathbf{1} \bullet \mathbf{x})}{(\mathbf{1} \bullet \mathbf{1})(\mathbf{x} \bullet \mathbf{x}) - (\mathbf{1} \bullet \mathbf{x})^2}.$$

Find this so called *line-of-best-fit* for the case where $n = 4$ and the given set of points is

$$\{(-1, 1), (1, -1), (3, -4), (5, -4)\}.$$

Hints for Problems

Problem Set 1

3. Write the third row as a linear combination of the other two.
5. The matrices in any counterexample cannot be square.

Problem Set 2

Express systems in matrix form. Reduce the matrix to *row echelon form* and then solve the corresponding reduced equation system by back-substitution, assigning free values to suitable unknowns. The operations in reduction to echelon form are: exchange or rows, multiplying a row by a non-zero number, and adding a multiple of one row to another.

4. Cramer's rule: for square non-singular matrix A the solution vector $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$ has entries $x_i = \frac{|A_i|}{|A|}$ where A_i is the matrix A with the ith column replaced by $\mathbf{b}$. Replacing $\mathbf{b}$ by each of the columns of the identity matrix allows computation of each of the columns of A^{-1} in turn.

6. In one direction, use Cramer's rule. For the converse use that $|AB| = |A| \cdot |B|$.

8. Suppose that there are two distinct solutions to the system and use them to construct infinitely many solutions to the corresponding homogeneous system. Then use Question 7.

Problem Set 3

5. $P\mathbf{x}$ is the form of the corresponding eigenvector.

Problem Set 4

Remember that a number is divisible by 3 exactly when the same is true of the sum of its digits.

Problem Set 5

3. Expand and write the resulting expression as the sum of squares.

Problem Set 6

1-5. For each problem, the columns of the required matrix are the images of the standard basis vectors under the linear transformation in question.

Problem Set 7

1. Find the rank of the matrix the rows of which are the given vectors.

2. Take a maximal independent set of rows from the echelon form of the matrix the rows of which are the given vectors.

4-5. For any such proof you must work with the definition of an independent set:

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_k\mathbf{u}_k = \mathbf{0} \rightarrow a_1 = a_2 = \cdots = a_k = 0.$$

6. Consider the original generating set and that which results from carrying out a row operation and show that each is contained in the span of the other. It then follows that each generate the same subspace.

Problem Set 8

6. Write $M\mathbf{u}$ as a linear combination of the columns of M.

7. Let M be the matrix the columns of which are the images of each of the standard basis vectors under L.

10. An independent set is never larger than a spanning set.

Problem Set 9

2. Use Question 7 of Set 8.

3. Remember the *Exchange Lemma* which implies that any independent set is no larger than any spanning set.

5. Think about extending a basis of the subspace to that of the whole space.

7. Take a basis for the intersection space and extend it to a bases for each of the containing spaces. Then consider the union of these bases.

10. Represent $U^{\perp}$ as the solution space of an homogeneous equation system.

Problem Set 10

4. Show inductively that the given set is orthogonal, and that all vectors are non-zero.

6. Row reduce the matrix consisting of the columns of **B** and of **C** in that order. The top three rows will then consist of I_3 and P, the change-of-basis matrix from **B**- to **C**-coordinates.

Answers to problems

Problem Set 1

1. $\begin{bmatrix} 2 & 4 \\ -2 & 2 \end{bmatrix}$, $\begin{bmatrix} 3 & 2 & -3 \\ -3 & 0 & 3 \\ -1 & -2 & 1 \end{bmatrix}$. 2. 12, 0. 3. $-(3,2,-3) - \frac{2}{3}(-3,0,3)$.4. $\frac{23}{2}$.
6. $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. 7. $\frac{1}{4}\begin{bmatrix} 1 & -3 & -1 \\ 1 & 1 & -1 \\ -1 & 3 & 5 \end{bmatrix}$. 8. $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. 9. 3. 10.

Problem Set 2

1. $(\frac{3}{2}, \frac{1}{2}, 1, 0, 0) + c_1(\frac{7}{2}, \frac{1}{2}, -3, 1, 0) + c_2(-\frac{1}{2}, \frac{1}{2}, 0, 0, 1)$. 2. $(1, 0, -1, 1) + c(-2, 1, 0, 0)$.
3. $\begin{bmatrix} 10 & -4 & 1 \\ -15 & 7 & -2 \\ 6 & -3 & 1 \end{bmatrix}$. 4. $(x, y, z) = (3, 2, 1)$.

Problem Set 3

1. -2 and 3. 2. $(2,1)^T$, $(3,2)^T$. 3. $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. 4.
$\begin{bmatrix} -6305 & 12738 \\ -12994 & 26052 \end{bmatrix}$.

Problem Set 4

1. $\mathbf{x_0} = (1,0,0)^T$, $\mathbf{x_1} = (2,1,1)$. 2. $x_n = 2x_{n-1} + y_n + z_n$, $y_n = x_{n-1} + 2y_{n-1} + z_{n-1}$, $z_n = x_{n-1} + y_{n-1} + 2z_{n-1}\,(n \geq 1)$.
3. $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ 4. $\lambda = 1, 4$. 5. $\lambda = 1 : (1,0,-1)^T$, $(0,1,-1)^T$ independent eigenvectors, $\lambda = 4 : (1,1,1)^T$ is an eigenvector.

6. $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ $P^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$.
7. $\frac{1}{3}\begin{bmatrix} 4^n+2 & 4^n-1 & 4^n-1 \\ 4^n-1 & 4^n+2 & 4^n-1 \\ 4^n-1 & 4^n-1 & 4^n+2 \end{bmatrix}$. 8. $\mathbf{x_n} = \frac{1}{3}\begin{bmatrix} 4^n+2 \\ 4^n-1 \\ 4^n-1 \end{bmatrix}$.

Problem Set 5

1. 6&7. $A = \begin{bmatrix} 5 & -5 \\ -5 & 1 \end{bmatrix}$

Problem Set 6

1. $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ $A^{-1} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$. 2. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. 3. $\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$. 4. $\mathrm{Rot}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$; $\mathrm{Rot}(\theta+\phi)$. 5. $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$. 6. $\begin{bmatrix} \cos(2\phi-\theta) & \sin(2\phi-\theta) \\ \sin(2\phi-\theta) & -\cos(2\phi-\theta) \end{bmatrix}$.
7. $\frac{\sqrt{6}-\sqrt{2}}{4}$, $\frac{\sqrt{6}+\sqrt{2}}{4}$.
8. $\begin{bmatrix} \cos(2\phi+\theta) & \sin(2\phi+\theta) \\ \sin(2\phi+\theta) & -\cos(2\phi+\theta) \end{bmatrix}$, $\mathrm{Ref}(\phi+\frac{\theta}{2})$. 9. $\mathrm{Rot}(2(\phi-\theta))$. 10. $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $\lambda = 1$, $\mathbf{x} = (1,1,1)^T$.

Problem Set 7

1. Independent. 2. $\{(1,-1,3,2),(0,1,-5,9),(0,0,1,2)\}$. 3. $\{(\frac{3}{2},1,0,0),(-2,0,1,0),(\frac{1}{2},0,0,1)\}$.
4.

Problem Set 10

3. $(\mathbf{v}\bullet\mathbf{v}_1, \mathbf{v}\bullet\mathbf{v}_2) = (-\sqrt{3}+\frac{3}{2}, 1+\frac{3\sqrt{3}}{2})$. 5. $\frac{1}{\sqrt{6}}(1,1,-2,0)$, $\frac{1}{\sqrt{66}}(-1,5,2,-6)$.

6. $\begin{bmatrix} 1 & 1 & 1 \\ -5 & -4 & -3 \\ 10 & 8 & 7 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -6 \\ 12 \end{bmatrix}$. 7. $\frac{1}{7}\begin{bmatrix} (-1)^n 5(a-b) + 6^n(2a+5b) \\ 2(-1)^{n+1}(a-b) + 6^n(2a+5b) \end{bmatrix}$. 10. $y = -\frac{9}{10}x - \frac{1}{5}$.