Mathematics 208 Classical Mechanics

Professor Peter M. Higgins

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This module on classical mechanics follows on and presumes the content of MA108 *Mechanics*. The central theme is that of rotating bodies and so we see in Sets 1 and 2 the focus is on problems involving the *centre of mass* and the *moment of inertia* of a system of particles and of a rigid body in two and three dimensions, including the *Perpendicular* and *Parallel axis theorems.* Sets 3 and 4 are based on energy and work considerations related to moments of inertia and feature standard problems involving the motion of masses on inclined planes, over pulley systems and the *torque* that results within systems subjected to external forces. Set 5 features questions involving both the *static* and *kinetic coefficient of friction* when forces are in play on objects moving over rough surfaces.

In the latter half of the module we introduce some new techniques apart from our standard approaches of the use of Newton's laws and Conservation of energy. Set 6 calls upon the technique of *Virtual work* to resolve forces on systems in equilibrium. Just as force is the rate of change of linear momentum, *torque* is the time derivative of *angular momentum* and these concepts are the work of Set 7. The *Euler-Lagrange equation* is introduced in Set 8 as an alternative to the Newtonian scheme in mechanics questions. Sets 9 and 10 have as their subject rotating frames of reference. *Coriolis forces* are explored in Set 9 while in Set 10 we return to the topic of *central forces.*

Solutions and Comments for the Problems

Problem Set 1

1. We put the weighted averages of the $m_k(x_k - \overline{x})$ to zero and solve for $\overline{x}$. (In doing so, we specify $\overline{x}$ to be the point M such that the torque of the system about M is 0):

$$\sum_k m_k(x_k - \overline{x}) = 0$$

$$\Rightarrow \sum_k m_k x_k - \overline{x}\sum_k m_k = 0$$

$$\therefore \overline{x} = \frac{\sum_k m_k x_k}{\sum_k m_k}.$$

Comment If we take the masses as distributed along a see-saw (line) then $\overline{x}$ is the position of the fulcrum that leaves the system in balance. The idea extends to two and three coordinates. Alternatively the *torque* (tendency to spin) around the centre of mass is 0. The calculation for this would be as above with masses replaced by forces that would simply involve the introduction of the local gravitational constant g in each term, which would cancel to give the same result.

2. Let $(\overline{x}, \overline{y})$ be the required coordinates. Then

$$\overline{x} = \frac{2\cdot 0 + 3a + 6a + 7\cdot 0}{2+3+6+7} = \frac{9a}{18} = \frac{a}{2}, \ \overline{y} = \frac{2\cdot 0 + 3\cdot 0 + 6a + 7a}{18} = \frac{13a}{18}$$

so that $(\overline{x}, \overline{y}) = (\frac{1}{2}a, \frac{13}{18}a)$.

3(a) In discrete notation we have

$$M\overline{x} = \sum_{i\in A} m_i x_i + \sum_{i \in B} m_i x_i = M_A\overline{x}_A + M_B\overline{x}_B.$$

$$\Rightarrow \overline{x}_A = \frac{M_A\overline{x}_A + M_B\overline{x}_B}{M}.$$

Comment We will extend this to more than two bodies and in the limiting case, to integrals, which are then written in the form:

$$M\overline{x} = \int x\,dm = \int x\frac{dm}{dx}\,dx = \int x\delta(x)\,dx$$

where $\delta(x)$ is the *density function* $\frac{dm}{dx}$.

(b) The shape is comprised of two rectangles. The first has mass $M_A = 1 \times (11-1) = 10$ and centre of mass at $(x_a, y_a) = (\frac{1}{2}, \frac{11}{2})$. The second has mass $M_B = 1 \times 5 = 5$ with centre of mass at $(\frac{5}{2}, \frac{1}{2})$. The total mass is $M = M_A + M_B = 10 + 5 = 15$. By part (a) the centre of mass of the entire body has coordinates:

$$(\overline{x}, \overline{y}) = (\frac{10(\frac{1}{2}) + 5(\frac{5}{2}), 10(\frac{11}{2}) + 5(\frac{1}{2})}{15}) = (\frac{35}{30}, \frac{115}{30}\Big) = (\frac{7}{6}, \frac{23}{6}\Big).$$

Comment Note that in this case the centre of mass lies outside the body. If the L were tossed into the air it would spin about this point.

4. Place the side of length a on the x-axis with the right angle at $(a, 0)$. The hypotenuse then has equation $y = \frac{b}{a}x$. Taking the density $\delta(x, y)$ of the triangle T to be 1 we then have:

$$M_x = \int_0^a \int_0^{\frac{bx}{a}} y \, dydx = \frac{1}{2}\int_0^a [y^2]_0^{\frac{bx}{a}} \, dx = \frac{1}{2}\int_0^a \frac{b^2x^2}{a^2} \, dx;$$

$$= \frac{1}{6}\frac{b^2}{a^2}[x^3]_0^a = \frac{b^2a}{6}.$$

$$M = \int_0^a y \, dx = \frac{b}{a}\int_0^a x \, dx = \frac{b}{2a}[x^2]_0^a = \frac{ba}{2};$$

$$\Rightarrow \overline{y} = \frac{b^2a}{6} \cdot \frac{2}{ba} = \frac{b}{3}.$$

By symmetry, $\overline{x} = a - \frac{a}{3}$, so that $(\overline{x}, \overline{y}) = (\frac{2a}{3}, \frac{b}{3})$, or, if we place the right-angle at the origin, we have the centre of mass of the triangle lies $(\frac{a}{3}, \frac{b}{3})$.

5. Alternatively we may use the formula:

$$M_y = \int_0^a xy \, dx = \frac{b}{a}\int_0^a x^2 \, dx = \frac{b}{3a}[x^3]_0^a = \frac{ba^3}{3a} = \frac{ba^2}{3}$$

$$\Rightarrow \overline{x} = \frac{M_y}{M} = \frac{ba^2}{3} \cdot \frac{2}{ba} = \frac{2a}{3},$$

yielding the same result as before.

And we may find $\overline{y}$ by integrating the limiting contribution from thin rectangular strips ydx to M_x. Each contribution is $\tilde{y} = \frac{y}{2}dx$ as the centre of mass of the strip lies at $(x, \frac{y}{2})$. This gives:

$$M_x = \int_0^a \frac{1}{2}y^2 \, dy = \frac{b^2}{2a^2}\int_0^a x^2 \, dx = \frac{b^2}{6a^2}[x^3]_0^a = \frac{b^2}{6a^2} \cdot a^3 = \frac{b^2a}{6}$$

$$\Rightarrow \overline{y} = \frac{M_x}{M} = \frac{b^2a}{6} \cdot \frac{2}{ba} = \frac{b}{3}.$$

6. We place the centre of the circle at the origin with the semicircle lying above. By symmetry we have $\overline{x} = 0$. For $\overline{y}$ we have the boundary equation $y = \sqrt{a^2 - x^2}$. Since y is an even function of x the integral becomes:

$$M_x = \frac{1}{2}\int_{-a}^a y^2 \, dx = \int_0^a (a^2 - x^2) \, dx = [a^2x - \frac{x^3}{3}]_0^a = [a^3 - \frac{a^3}{3}] = \frac{2a^3}{3}.$$

$$M = \int_{-a}^{a} y\,dx = \int_{-a}^{a} \sqrt{a^2 - x^2}\,dx = 2\int_{0}^{a} \sqrt{a^2 - x^2}dx$$

Put $x = a\sin\theta$ so $dx = a\cos\theta\,d\theta$, $\sqrt{a^2 - x^2} = a\cos\theta$; $x = a \mapsto \theta = \frac{\pi}{2}$, $x = 0 \mapsto \theta = 0$,

$$M = a^2\int_0^{\frac{\pi}{2}} 2\cos^2\theta\,d\theta = a^2\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)\,d\theta = \frac{\pi a^2}{2}.$$

Hence

$$\overline{y} = \frac{M_x}{M} = \frac{2a^3}{3}\cdot\frac{2}{\pi a^2} = \frac{4a}{3\pi}.$$

Therefore $(\overline{x}, \overline{y}) = (0, \frac{4a}{3\pi})$.

Comment We have essentially solved this problem in MA108 using the *Theorems of Pappus*.

7. By symmetry $\overline{x} = 0$. To calculate $\overline{y}$ we model the mass contribution using vertical strips, the centre of mass of which corresponds to $(\tilde{x}, \tilde{y}) = (x, \frac{4-x^2}{2})$. The moment of the strip about the x-axis is:

$$M_x = \int \tilde{y}\,dm = \int_{-2}^{2} \frac{\delta}{2}(4 - x^2)^2\,dx$$

$$\int_{-2}^{2} \frac{\delta}{2}(4-x^2)^2\,dx = \delta\int_0^2 (16-8x^2+x^4) = \delta[16x - \frac{8}{3}x^3 + \frac{x^5}{5}]_0^2 = \delta[32 - \frac{64}{3} + \frac{32}{5}]$$

$$= \delta\frac{480 - 320 + 96}{15} = \frac{256}{15}\delta.$$

$$M = \int dm = \int_{-2}^{2} \delta(4-x^2)\,dx = 2\delta\int_0^2 (4-x^2)\,dx = 2\delta[4x - \frac{x^3}{3}]_0^2 = 2\delta[8 - \frac{8}{3}] = \frac{32}{3}\delta.$$

$$\Rightarrow \overline{y} = \frac{M_x}{M} = \frac{256}{15}\cdot\frac{3}{32} = \frac{8}{5};$$

$$\therefore (\overline{x}, \overline{y}) = (0, \frac{8}{5}).$$

8. Replacing δ in the previous problem by $\delta = 2x^2$ we obtain:

$$M_x = \int_{-2}^{2} x^2(4 - x^2)^2\,dx = 2\int_0^2 (16x^2 - 8x^4 + x^6)\,dx = 2[\frac{16x^3}{3} - \frac{8x^5}{5} + \frac{x^7}{7}]_0^2$$

$$= 2(\frac{128}{3} - \frac{256}{5} + \frac{128}{7}) = 2(\frac{128(35+15) - 256(21)}{105}) = 256(\frac{50 - 42}{105}) = \frac{2048}{105}.$$

$$M = \int_{-2}^{2} 2x^2(4 - x^2)\,dx = 2\int_0^2 (8x^2 - 2x^4)\,dx = 4[\frac{4x^3}{3} - \frac{x^5}{5}]_0^2$$

$$= 4(\frac{32}{3} - \frac{32}{5}) = 128(\frac{5-3}{15}) = \frac{256}{15}.$$

$$\Rightarrow \overline{y} = \frac{M_x}{M} = \frac{2048}{105} \cdot \frac{15}{256} = \frac{8}{7}$$

$$\therefore (\overline{x}, \overline{y}) = (0, \frac{8}{7}).$$

9.

$$M = \int_0^1 \int_0^{2x} \delta(x,y)\, dydx = 6 \int_0^1 \int_0^{2x} (x+y+1)\, dydx = 6 \int_0^1 [xy + \frac{y^2}{2} + y]_0^{y=2x}\, dx$$

$$= 6 \int_0^1 (2x^2 + 2x^2 + 2x)\, dx = 12 \int_0^1 (2x^2 + x)\, dx = 12[\frac{2x^3}{3} + \frac{x^2}{2}]_0^1$$

$$= 12(\frac{2}{3} + \frac{1}{2}) = \frac{12 \times 7}{6} = 14.$$

$$M_x = \int_0^1 \int_0^{2x} y\delta(x,y)\, dydx = \int_0^1 \int_0^{2x} (6xy + 6y^2 + 6y)\, dydx$$

$$= \int_0^1 [3xy^2 + 2y^3 + 3y^2]_0^{y=2x}\, dx = \int_0^1 (28x^3 + 12x^2)\, dx$$

$$= [7x^4 + 4x^3]_0^1 = 7 + 4 = 11.$$

$$M_y = \int_0^1 \int_0^{2x} x\delta(x,y)\, dydx = \int_0^1 \int_0^{2x} (6x^2 + 6xy + 6x)\, dydx$$

$$= \int_0^1 [6x^2y + 3xy^2 + 6xy]_0^{y=2x}\, dx = \int_0^1 (24x^3 + 12x^2)\, dx$$

$$= [6x^4 + 4x^3]_0^1 = 6 + 4 = 10.$$

$$\overline{x} = \frac{M_y}{M} = \frac{10}{14} = \frac{5}{7},\ \overline{y} = \frac{M_x}{M} = \frac{11}{14}.$$

$$\therefore (\overline{x}, \overline{y}) = (\frac{5}{7}, \frac{11}{14}).$$

10.

$$I_x = \int_0^1 \int_0^{2x} y^2\delta(x,y)\, dydx = \int_0^1 \int_0^{2x} (6xy^2 + 6y^3 + 6y^2)\, dydx$$

$$= \int_0^1 [2xy^3 + \frac{3}{2}y^4 + 2y^3]_0^{y=2x}\, dx = \int_0^1 (40x^4 + 16x^3)\, dx = [8x^5 + 4x^4]_0^1 = 8 + 4 = 12.$$

$$I_y = \int_0^1 \int_0^{2x} x^2\delta(x,y)\, dydx = \int_0^1 \int_0^{2x} (6x^3 + 6x^2y + 6x^2)\, dydx$$

$$= \int_0^1 [6x^3y + 3x^2y^2 + 6x^2y]_0^{y=2x}\, dx = \int_0^1 (24x^4 + 12x^3)\, dx$$
$$= [24\frac{x^5}{5} + 3x^4]_0^1 = \frac{24}{5} + 3 = \frac{39}{5}.$$
$$I_0 = I_x + I_y = 12 + \frac{39}{5} = \frac{99}{5}.$$
$$R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{12}{14}} = \sqrt{\frac{6}{7}},\ R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{39/5}{14}} = \sqrt{\frac{39}{70}},\ R_0 = \sqrt{\frac{I_0}{M}} = \sqrt{\frac{99/5}{14}} = \sqrt{\frac{99}{70}}.$$

Problem Set 2

1.
$$M = \int_0^1 \int_{x^2}^x dydx = \int_0^1 (x - x^2)\, dx = [\frac{x^2}{2} - \frac{x^3}{3}]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$
$$M_y = \int_0^1 \int_{x^2}^x y\, dydx = \int_0^1 [\frac{y^2}{2}]_{x^2}^x\, dx = \int_0^1 (\frac{x^2}{2} - \frac{x^4}{2})\, dx = [\frac{x^3}{6} - \frac{x^5}{10}]_0^1$$
$$= \frac{1}{6} - \frac{1}{10} = \frac{5-3}{30} = \frac{2}{30} = \frac{1}{15},$$
$$M_x = \int_0^1 \int_{x^2}^x x\, dydx = \int_0^1 [xy]_{y=x^2}^{y=x}\, dx = \int_0^1 (x^2 - x^3)\, dx = [\frac{x^3}{3} - \frac{x^4}{4}]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$
$$(\overline{x}, \overline{y}) = (\frac{1/12}{1/6}, \frac{1/15}{1/6}) = (\frac{1}{2}, \frac{1}{5}).$$

2. Since $y^2 + z^2$ is an even function in all three variables we have:

$$I_x = \int_{-\frac{c}{2}}^{\frac{c}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (y^2 + z^2)\delta\, dxdydz = 8\delta \int_0^{\frac{c}{2}} \int_0^{\frac{b}{2}} \int_0^{\frac{a}{2}} (y^2 + z^2)\, dxdydz$$

$$= 4a\delta \int_0^{\frac{c}{2}} \int_0^{\frac{b}{2}} (y^2 + z^2)\, dydz = 4a\delta \int_0^{\frac{c}{2}} [\frac{y^3}{3} + z^2y]_0^{\frac{b}{2}}\, dz = 4a\delta \int_0^{\frac{c}{2}} (\frac{b^3}{24} + \frac{z^2b}{2})\, dz$$
$$= 4a\delta[\frac{b^3z}{24} + \frac{z^3b}{6}]_0^{\frac{c}{2}} = 4a\delta(\frac{b^3c}{48} + \frac{c^3b}{48}) = \frac{abc\delta}{12}(b^2 + c^2) = \frac{M}{12}(b^2 + c^2).$$

Comment By symmetry, the values of I_y and I_z can each written down as the above calculation solves all three problems up to the naming of the variables.

3. By symmetry $\overline{x} = \overline{y} = 0$. To find $\overline{z}$ we may put $\delta = 1$ and first find M, which will equal the volume of the solid:

$$M = \int\int_D \int_0^{z=4-x^2-y^2} dzdydx = \int\int_D (4 - x^2 - y^2)\, dydx.$$

Switch to polar coordinates:

$$= \int_0^{2\pi} \int_0^2 (4-r^2) r \, d\theta dr = 2\pi \int_0^2 (4r - r^3)\, dr$$

$$= 2\pi[2r^2 - \frac{r^4}{4}]_0^2 = 2\pi(8-4) = 8\pi.$$

$$M_{xy} = \int\int_D \int_0^{4-x^2-y^2} z\, dzdydx = \int\int_D [\frac{z^2}{2}]_0^{4-x^2-y^2} dydx$$

$$= \frac{1}{2}\int\int_D (4-x^2-y^2)^2\, dydx$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^2 (4-r^2)^2 r\, d\theta dr = \pi \int_0^2 (4-r^2)^2 r\, dr,$$

put $u = 4 - r^2$ so that $du = -2rdr$ and $rdr = -\frac{1}{2}du$ and we obtain

$$-\frac{\pi}{2}\int_4^0 u^2\, du = \frac{\pi}{2}[\frac{u^3}{3}]_0^{u=4} = \frac{\pi}{2}\cdot\frac{64}{3} = \frac{32\pi}{3}.$$

$$\Rightarrow \overline{z} = \frac{M_{xy}}{M} = \frac{32\pi}{3(8\pi)} = \frac{4}{3}.$$

$$\therefore (\overline{x}, \overline{y}, \overline{z}) = (0, 0, \frac{4}{3}).$$

4. We simply note that

$$I_z = \int r^2\, dm = \int (x^2+y^2)\, dm = \int x^2\, dm + \int y^2\, dm = I_x + I_y.$$

5. We may assume that the perpendicular distance between the axes lies along the x-axis and that the centre of mass lies at the origin. The moment of inertia relative to the z-axis is

$$I_{cm} = \int r^2\, dm = \int (x^2+y^2)\, dm$$

On the other hand the moment of inertia relative to the z' axis is

$$I = \int ((x+d)^2 + y^2)\, dm = \int (x^2+y^2)\, dm + d^2 \int dm + 2d \int x\, dm.$$

The first term is I_{cm}, the moment of inertia about the centre of mass, while the second equals md^2. The final term is a multiple $(2d)$ of the x-coordinate of the centre of mass, which is 0 as the centre of mass lies at the origin.

Therefore we obtain the Parallel axis theorem:

$$I = I_{cm} + md^2.$$

6(a) Let Q denote the space occupied by the object (as R often denotes a relevant radius). Then we have:

$$I = \int\int\int_Q \rho x^2\, dV = \int_{-\frac{l}{2}}^{\frac{l}{2}} \rho x^2 s\, dx = 2\rho s \int_0^{\frac{l}{2}} x^2\, dx = 2\rho s[\frac{x^3}{3}]_0^{\frac{l}{2}} = 2\rho s\frac{l^3}{24} = \frac{\rho sl}{12} l^2$$

$$I = \frac{ml^2}{12}.$$

(b) By part (a) we have:

$$I = \frac{ml^2}{12} + m(\frac{l}{2})^2 = \frac{m}{12}(l^2 + 3l^2) = \frac{4ml^2}{12} = \frac{ml^2}{3}.$$

7(a) The volume element of the integration is $dV = lrdrd\theta$

$$I = \int\int\int_Q \rho r^2\, dV = l\rho \int_0^{2\pi}\int_0^R r^3\, drd\theta = 2\pi l\rho[\frac{R^4}{4}]_0^{r=R}$$

$$= \frac{2\pi l\rho R^4}{4} = \frac{R^2}{2}(\rho\pi R^2 l) = \frac{1}{2}mR^2.$$

7(b) By Questions 5 and part (a) we obtain by symmetry that:

$$I_z = I_x + I_y = 2I_x$$

$$\Rightarrow I_x = \frac{1}{2}(\frac{1}{2}mR^2) = \frac{1}{4}mR^2.$$

(c) The contribution from a thin disc of height dz is by part (b) and the Parallel axis theorem equal to $m\frac{dz}{h}(\frac{R^2}{4} + z^2)$. Hence

$$I = \frac{m}{4h}\int_{-\frac{h}{2}}^{\frac{h}{2}} (R^2 + 4z^2)\, dz = \frac{m}{4h}(R^2 h + 8\int_0^{\frac{h}{2}} z^2\, dz)$$

$$= \frac{mR^2}{4} + \frac{2m}{3h}[z^3]_0^{\frac{h}{2}} = \frac{mR^2}{4} + \frac{2m}{3h}\cdot\frac{h^3}{8} = \frac{m}{12}(3R^2 + h^2).$$

8(a)

$$I_x = \int\int_A y^2\, dA = \int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}} y^2\, dxdy = a\int_{-\frac{b}{2}}^{\frac{b}{2}} y^2\, dy = 2a\int_0^{\frac{b}{2}} y^2\, dy = \frac{2a}{3}[y^3]_0^{\frac{b}{2}} = \frac{2ab^3}{24} = \frac{mb^2}{12},$$

as $ab = m$.

(b) By the Parallel axis theorem and part (a) we obtain:

$$I = \frac{mb^2}{12} + m\left(\frac{b}{2}\right)^2 = mb^2(\frac{1}{12} + \frac{1}{4}) = \frac{mb^2}{3}.$$

9(a) By the Perpendicular axis theorem and Question 8(a) we obtain:

$$I_z = \frac{m}{12}(a^2 + b^2).$$

(b) By the Perpendicular axis theorem and part (a) we obtain:

$$I = I_z + m\big((\frac{a}{2})^2 + (\frac{b}{2})^2\big) = m(\frac{a^2}{12} + \frac{b^2}{12} + \frac{a^2}{4} + \frac{b^2}{4}) = \frac{m}{3}(a^2 + b^2).$$

Comment: Alternatively the answer is $\frac{1}{4}$ of the moment of inertia about an axis perpendicular to the centre of a plate of double the dimensions, which by part (a) is:

$$\frac{1}{4} \cdot \frac{(2a)(2b)}{12}((2a)^2 + (2b)^2) = \frac{m}{12} \cdot 4(a^2 + b^2) = \frac{m}{3}(a^2 + b^2).$$

(c) Again by the Perpendicular axis theorem and part (a) we obtain:

$$I = I_z + m(\frac{a}{2})^2 = m(\frac{a^2}{12} + \frac{b^2}{12} + \frac{a^2}{4}) = \frac{m}{12}(4a^2 + b^2).$$

Comment Alternatively the answer is $\frac{1}{2}$ of the moment of inertia about an axis perpendicular to the centre of a plate of double the dimension in the x direction, which by part (a) is:

$$\frac{1}{2} \cdot \frac{2ab}{12}\big((2a)^2 + b^2\big) = \frac{m}{12}(4a^2 + b^2).$$

10. We have $\frac{dm}{dz} = \rho\pi r^2$ so $dm = \rho\pi r^2 dz$. The mass m of our cone is then given by $m = \rho V = \frac{\pi\rho}{3}\rho R^2 h$ so that $\rho = \frac{3m}{\pi R^2 h}$. Hence

$$dm = \frac{3M}{\pi R^2 h} \cdot \pi r^2 dz = \frac{3Mr^2}{R^2 h} dz.$$

However, by similar triangles we have $\frac{r}{R} = \frac{z}{h}$. Substituting accordingly gives:

$$dm = \frac{3Mz^2}{h^3}\, dz.$$

The moment I about the z-axis then satisfies:

$$\frac{dI}{dm} = \frac{1}{2}r^2 \Rightarrow dI = \frac{1}{2}r^2 dm = \frac{1}{2}r^2 \cdot \frac{3Mz^2}{h^3} dz = \frac{3mz^2}{2h^3} \cdot \frac{R^2 z^2}{h^2} dz$$

$$\Rightarrow I = \frac{3mR^2}{2h^5} \int_0^h z^4\, dz = \frac{3mR^2}{10h^5}[z^5]_0^h = \frac{3mR^2}{10h^5} \cdot h^5 = \frac{3}{10} mR^2.$$

Problem Set 3

1(a)

$$dI = \frac{1}{2}r^2 dm = \frac{1}{2}r^2(\rho\pi r^2)dx = \frac{1}{2}\rho\pi r^4 dx = \frac{1}{2}\rho\pi(R^2 - x^2)^2 dx$$

$$\Rightarrow I = \frac{1}{2}\rho\pi \int_{-R}^{R} (R^2 - x^2)^2 dx = \rho\pi \int_0^R (R^4 - 2x^2R^2 + x^4)\, dx$$

$$= \rho\pi[R^4 x - \frac{2}{3}R^2x^3 + \frac{x^5}{5}]_0^R = \rho\pi R^5[1 - \frac{2}{3} + \frac{1}{5}] = \frac{8}{15}\rho\pi R^5.$$

Now $m = \frac{4}{3}\pi R^3 \rho$ so we obtain

$$I = (\frac{4}{3}\pi\rho R^3) \cdot \frac{2}{5}R^2 = \frac{2}{5}mR^2.$$

(b) Let m be the mass of the hollow sphere of density ρ with respective radii R_1 and R_2. We obtain

$$I = \frac{8}{15}\rho\pi R_1^5 - \frac{8}{15}\rho\pi R_2^5 = \frac{8}{15}\rho\pi(R_1^5 - R_2^5).$$

Now

$$\rho = \frac{m}{V} = \frac{m}{\frac{4}{3}\pi(R_1^3 - R_2^3)}$$

$$\Rightarrow I = \frac{8\pi}{15} \cdot \frac{3}{4}\frac{m(R_1^5 - R_2^5)}{\pi(R_1^3 - R_2^3)} = \frac{2m}{5} \cdot \frac{R_1^5 - R_2^5}{R_1^3 - R_2^3}.$$

2. Write $R_1 = R = R_2 + r$. Then we require

$$\frac{2m}{5} \lim_{r\to 0} \frac{(R_2 + r)^5 - R_2^5}{(R_2 + r)^3 - R_2^3} = \frac{2m}{5} \lim_{r\to 0} \frac{5R_2^4 r + o(r)}{3R_2^2 r + o(r)}$$

$$= \frac{2m}{5} \lim_{r\to 0} \frac{5R_2^4 + \frac{o(r)}{r}}{3R_2^2 + \frac{o(r)}{r}} = \frac{2m}{5} \cdot \frac{5R^4 + 0}{3R^2 + 0}$$

$$= \frac{2m}{5} \cdot \frac{5R^2}{3} = \frac{2}{3}mR^2.$$

3(a) The velocity of a single point particle rotating at a distance r with angular ω is $r\omega$. Hence the kinetic energy of that particle of mass m is $\frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2$. Hence if we sum this over a discrete set of particles of mass m_i at a distance r_i from the axis of rotation we obtain:

$$E = \sum_i \frac{1}{2}m_i r_i^2 \omega^2 = \frac{1}{2}\omega^2 \sum_i m_i r_i^2 = \frac{1}{2}I\omega^2.$$

(b) In the continuous case the sum takes the form of the integral:

$$\frac{1}{2}\omega^2 \int_V r^2\,dm = \frac{1}{2}I\omega^2.$$

4 & 5. In each case we have the energy equation:

$$mgh = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 = \frac{1}{2}I(\frac{v}{R})^2 + \frac{1}{2}mv^2.$$

For a solid sphere, $I = \frac{2}{5}mR^2$ so we obtain:

$$mgh = \frac{m}{5}v^2 + \frac{1}{2}mv^2$$

$$\Rightarrow gh = (\frac{1}{5} + \frac{1}{2})v^2 = \frac{7v^2}{10}$$

$$\therefore v = \sqrt{\frac{10gh}{7}}.$$

For a thin spherical shell of the same mass the calculation is identical except that the coefficient of $\frac{2}{5}$ is replaced by $\frac{2}{3}$, hence we get

$$gh = (\frac{1}{3} + \frac{1}{2})v^2 = \frac{5}{6}v^2$$

$$\therefore v = \sqrt{\frac{6gh}{5}}.$$

6(a) We have $I = \frac{1}{2}mR^2$. Hence the coefficient of v^2 is $\frac{1}{4} + \frac{1}{2} = \frac{3}{4}$ giving $v^2 = \frac{4}{3}gh$ and so $v = \sqrt{\frac{4gh}{3}}$.

(b) Since $\frac{6}{5} < \frac{4}{3} < \frac{10}{7}$ (as $18 < 20$ and $28 < 30$) it follows that v is greater for the solid sphere as compared to the cylinder, which in turn will beat the spherical shell.

7. Let m be the mass of the hoop, so that $I = mR^2$. Let v be the velocity of the hoop at the bottom of the hill so for a rolling hoop we have $v = R\omega$, where ω is the angular velocity of the hoop at this point. Equating the gain in kinetic energy with the loss in potential energy yields:

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh$$

$$\Rightarrow mv^2 + mR^2\left(\frac{v}{R}\right)^2 = 2mgh$$

$$\Rightarrow 2v^2 = 2gh$$

$$\therefore v = \sqrt{gh}.$$

Comment: note the result only depends on h (and g) and not on the mass or radius of the hoop. Also the kinetic energy of the hoop is equally shared between rotational energy and its speed. The hoop, with all its mass on the perimeter, is slower that all the 3D objects of Questions 4-6. An actual race between the four rolling objects can be viewed at https://en.wikipedia.org/wiki/Moment_of_inertia

8. This follows at once by changing variables in the integral:

$$\int\int_R x\, dA = \int\int_R x\, dxdy = \int\int_R (r\cos\theta)r\, drd\theta = \int\int_R r^2\cos\theta\, drd\theta$$

and similarly $\int\int_R y\, dA = \int\int_R r\sin\theta\, drd\theta$.

9. First we calculate the area of the cardiod:

$$A = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a(1+\sin\theta)} r\, drd\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [r^2]_0^{a(1+\sin\theta)}\, d\theta$$

$$= a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+2\sin\theta+\sin^2\theta)\, d\theta = a^2\big(\pi + \int_{0-\frac{\pi}{2}}^{\frac{\pi}{2}} (\frac{1-\cos 2\theta}{2})\, d\theta\big)$$

$$= a^2\big(\pi + \frac{\pi}{2}\big) = \frac{3\pi a^2}{2}.$$

Comment Note that the integral involving $\sin\theta$ is 0 because $\sin\theta$ is odd and the limits are symmetric about 0; the period of $\cos 2\theta$ is π so its integral over an interval of length π is also 0.

By symmetry, $\overline{x} = 0$ as $r(\theta) = r(\pi - \theta)$ so the cardiod is symmetric in the y-axis. Again by the same symmetry, $\overline{y}$ for the cardiod is the same as for that portion of the cardiod in the 4th and 1st quadrants, the mass of which is $\frac{3\pi a^2}{4}$.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a(1+\sin\theta)} r^2\sin\theta\, drd\theta = \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta[r^3]_0^{a(1+\sin\theta)}\, d\theta$$

$$= \frac{a^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\sin\theta)^3\sin\theta\, d\theta$$

Expanding the integrand gives $\sin\theta + 3\sin^2\theta + 3\sin^3\theta + \sin^4\theta$. The first and third terms are odd functions which integrate to 0. For the second term we have an even function and so the contribution is:

$$2\int_0^{\frac{\pi}{2}} \frac{1-\cos 2\theta}{2}\, d\theta = \frac{\pi}{2} - \frac{1}{2}[\sin 2\theta]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - \frac{1}{2}(-1-0) = \frac{\pi+1}{2}.$$

Now

$$\sin^4\theta = \big(\frac{1-\cos 2\theta}{2}\big)^2 = \frac{1}{4}\big(1 - 2\cos 2\theta + \cos^2 2\theta\big) \text{ which contributes:}$$

$$\frac{1}{2}\big(\frac{\pi}{2} - [\sin 2\theta]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{1+\cos 4\theta}{2}\, d\theta\big) = \frac{1}{2}\big(\frac{\pi}{2}+1+\frac{\pi}{4}\big) = \frac{1}{2}\big(\frac{2\pi+4+\pi}{4}\big) = \frac{3\pi+4}{8}.$$

$$\therefore \overline{y} = \frac{a^3}{3} \cdot \frac{3\pi + 4}{8} \cdot \frac{4}{3\pi a^2} = \frac{(3\pi + 4)a}{18\pi}.$$

10. By symmetry we have $r(\theta) = r(\frac{\pi}{2} - \theta)$ so we have symmetry with respect to the line $\theta = \frac{\pi}{4}$. It follows that $\overline{x} = \overline{y}$. Now

$$A = \int_0^{\frac{\pi}{2}} \int_0^{\sin 2\theta} r\, dr d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} [r^2]_0^{\sin 2\theta}\, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^2 2\theta\, d\theta = \frac{1}{4} \int_0^{\frac{\pi}{2}} (1 - \cos 4\theta)\, d\theta = \frac{\pi}{8}.$$

$$\int_0^{\frac{\pi}{2}} \int_0^{\sin 2\theta} r^2 \cos\theta\, dr d\theta = \frac{1}{3} \int_0^{\frac{\pi}{2}} [r^3]_0^{\sin 2\theta} \cos\theta\, d\theta = \frac{1}{3} \int_0^{\frac{\pi}{2}} \sin^3 2\theta \cos\theta\, d\theta$$

$$= \frac{8}{3} \int_0^{\frac{\pi}{2}} \sin^3\theta \cos^4\theta\, d\theta$$

Put $u = \cos\theta$, whence $-du = \sin\theta\, d\theta$, and when $\theta = 0$, $u = 1$, $\theta = \frac{\pi}{2}$, $u = 0$ so our integral becomes:

$$-\frac{8}{3} \int_1^0 (1 - u^2) u^4\, du = \frac{8}{3} \int_0^1 (u^4 - u^6)\, du = \frac{8}{3} [\frac{u^5}{5} - \frac{u^7}{7}]_0^1$$

$$= \frac{8}{3} (\frac{1}{5} - \frac{1}{7}) = \frac{8}{3} \cdot \frac{7 - 5}{35} = \frac{16}{105}.$$

Hence

$$\overline{x} = \overline{y} = \frac{16}{105} \cdot \frac{8}{\pi} = \frac{128}{105\pi}.$$

Problem Set 4

1. The work done in joules is:

$$W = \tau \Delta\theta = 50 \times 60 \times 2\pi = 600\pi = 1884 \cdot 96 J.$$

The mean power in watts is then:

$$P = \frac{W}{t} = \frac{600\pi}{12} = 50\pi = 157 \cdot 08\, \text{watts}.$$

2(a) The torque τ on the pulley is provided by the tension T in the string so that $\tau = TR$, where R is the radius of the pulley (which is not given in the problem, but never mind, press on). Also $\tau = I\alpha$ where the angular acceleration α satisfies $\alpha = \frac{a}{R}$, where a is the magnitude of the acceleration of a point on the circumference of the pulley, which is the same as that of the falling mass. Using $\tau = I\alpha$, this all yields:

$$TR = \frac{1}{2} M R^2 \alpha = \frac{M R^2}{2} \cdot \frac{a}{R}$$

$$\Rightarrow T = \frac{1}{2}Ma.$$

(b) Applying Newton's Law to the mass m we obtain:

$$mg - T = ma$$
$$\Rightarrow mg - \frac{Ma}{2} = ma$$
$$\Rightarrow a(m + \frac{M}{2}) = mg$$
$$\therefore a = \frac{2mg}{2m+M}.$$

(c) Since the acceleration is constant we may use the SUVAT equation $v^2 = u^2 + 2as$: indeed since $u = 0$ we get immediately that

$$v^2 = \frac{4mgh}{2m+M}$$
$$\therefore v = 2\sqrt{\frac{mgh}{2m+M}}.$$

3. Let v denote the velocity of the mass upon hitting the ground and ω the angular velocity of the pulley at the same moment. Then by conservation of energy and the fact that the moment of inertia of the pulley is $\frac{1}{2}MR^2$ we obtain:

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh$$
$$\Rightarrow 2mv^2 + MR^2\omega^2 = 4mgh. \qquad (1)$$

At the same time we have $v = R\omega$ so that (1) may be written as:

$$2mR^2\omega^2 + MR^2\omega^2 = 4mgh$$
$$\Rightarrow \omega^2(2m+M)R^2 = 4mgh$$
$$\therefore \omega = \frac{2}{R}\sqrt{\frac{mgh}{2m+M}} \text{ and so } v = 2\sqrt{\frac{mgh}{2m+M}}.$$

4(a) By Newton's law we have $T_1 = m_1 a$ and $m_2 g - T_2 = m_2 a$.

(b) We apply $\tau = I\alpha$ to the pulley, and this takes the form

$$\Rightarrow (T_2 - T_1)R = \frac{1}{2}MR^2 \cdot \frac{a}{R} = \frac{MRa}{2}$$
$$\therefore T_2 - T_1 = \frac{Ma}{2} \qquad (2)$$

Substituting from (a) accordingly into (2) then gives:

$$m_2 g - m_2 a - m_1 a = \frac{Ma}{2}$$

$$\Rightarrow a(\frac{M}{2} + m_2 + m_1) = m_2 g$$

$$\therefore a = \frac{2m_2 g}{2m_1 + 2m_2 + M}.$$

5(a) We again have $m_2 g - T_2 = m_2 a$. However Newton's law applied to m_1 now gives $T_1 - m_1 g \sin\theta - m_1 g \mu \cos\theta = m_1 a$. Equation (2) holds as before. Substituting now gives:

$$m_2 g - m_2 a - m_1 a - m_1 g \sin\theta - m_1 g \mu \cos\theta = \frac{Ma}{2}$$

$$\Rightarrow a(\frac{M}{2} + m_2 + m_1) = m_2 g - m_1 g(\sin\theta + \mu\cos\theta)$$

$$\therefore a = \frac{2m_2 - 2m_1(\sin\theta + \mu\cos\theta)}{M + 2m_1 + 2m_2} g.$$

(b) The system will be in equilibrium when $a = 0$, which is to say when $m_2 = m_1(\sin\theta + \mu\cos\theta)$.

Comment Of course if m_2 is less than this value then m_1 will slide down the slope, pulling m_1 upwards.

6(a) The retarding force, F_μ due to friction is $\mu N = \mu m g \cos\theta$. Hence $F = ma$ gives

$$mg\sin\theta - mg\mu\cos\theta = ma$$

$$\therefore a = g\sin\theta - g\mu\cos\theta. \tag{3}$$

(b) On the other hand $\tau = I\alpha$ again gives that $F_\mu = \frac{ma}{2}$ whence

$$mg\mu\cos\theta = \frac{ma}{2}$$

$$\therefore \mu = \frac{a}{2g\cos\theta}. \tag{4}$$

(c) Substituting from (4) into (3) for μ then yields:

$$a = g\sin\theta - g\cos\theta \cdot \frac{a}{2g\cos\theta}$$

$$\therefore a = \frac{2}{3} g\sin\theta.$$

(d) Substituting this answer back into (4) now gives

$$\mu = \frac{2g\sin\theta}{3 \cdot 2g\cos\theta} = \frac{1}{3}\tan\theta.$$

It follows that the frictional coefficient must be able to attain this value, or in other words, for a given value of μ, the maximum angle that will allow pure rolling is

$$\theta_{\max} = \arctan(3\mu).$$

7. As the beam is in equilibrium, we may equate the torque about the suspension point, which we take as the origin, to 0. The endpoints of the beam then lie at $x = -a$ and $x = L - a$. The centre of mass of the beam is then at its midpoint, which lies at

$$x = \frac{(L-a) + (-a)}{2} = \frac{L - 2a}{2}.$$

Equating the torque about the origin to 0 we get:

$$(L-a)m = (\frac{L-2a}{2})M$$

$$\therefore M = \frac{2m(L-a)}{L-2a} = 2m\big(1 + \frac{a}{L-2a}\big).$$

Comment Note that $L = 2a$ is only physically possible if $m = 0$. However, although $a < L$ it is certainly possible that $L - 2a < 0$ and $M \to 0$ as $a \to L$.

8(a) By Newton's law we have

$$mg - T = ma$$

where T is the tension in the string. Equally $\tau = I\alpha$ becomes

$$Tr = \frac{1}{2}mR^2\frac{a}{r} \Rightarrow T = \frac{ma}{2}(\frac{R}{r})^2$$

$$\text{Hence } mg = \frac{ma}{2}(\frac{R}{r})^2 + ma$$

$$\Rightarrow a(1 + \frac{1}{2}(\frac{R}{r})^2) = g$$

$$\therefore a = \frac{2g}{2 + \left(\frac{R}{r}\right)^2}.$$

(b) Clearly a increases with r so the maximum acceleration is realised by putting $r = R$, in which case $a = \frac{2}{3}g$.

9. Taking the upward direction as positive, we have the net torque at A is zero as the system is static. By considering the torques acting at A we obtain:

$$LR_B - mga = 0 \Rightarrow R_B = \frac{a}{L} \cdot mg.$$

By symmetry we get $R_A = mg\frac{L-a}{L} = (1 - \frac{a}{L})mg$.

10. For equilibium the net force on the piston head must be zero. The radius R of the torque arm satisfies $\sin\theta = \frac{d}{R}$ so that $R = \frac{d}{\sin\theta}$. The force F exerted by the torque of the arm is perpendicular to the lever arm. Its component in the direction of the piston shaft is $F\cos(\frac{\pi}{2} - \theta) = F\sin\theta$. Now $F\sin\theta - P = 0$ so that $F = \frac{P}{\sin\theta}$. Hence we obtain:

$$M = RF = \frac{d}{\sin\theta} \cdot \frac{P}{\sin\theta} = \frac{Pd}{\sin^2\theta}.$$

Comment We shall solve this problem again as Question 6 of Set 6 by the method of Virtual work.

Problem Set 5

1. In the static case we have $F_1 = \mu_s N$, where $N = 100\text{N}$, the weight of the block. Hence we get $40 = \mu_s 100$ so that $\mu_s = 0 \cdot 4$.

For the coefficient of kinetic friction, the force needed to maintain a constant velocity was $20N$. Hence $F_2 = \mu_k N$ so that $20 = 100\mu_k$ and so $\mu_k = 0 \cdot 2$.

2(a) Since the vertical forces are in equilibrium the upward force on the car is $R = 1200g$. Since the car is skidding $F = \mu R = 0 \cdot 8 \times 1200g = 9408\text{N}$. Now $F = ma = 1200a$ so that

$$1200a = -9408 \Rightarrow a = -7 \cdot 84\,\text{m/sec}^2.$$

(b) Using $v^2 = u^2 + 2as$ we obtain

$$0^2 = 20^2 + 2(-7 \cdot 84)s$$

$$\Rightarrow s = \frac{400}{15 \cdot 68} = 25 \cdot 5\,\text{m}\,(3\,\text{s.f.}).$$

(c) Using $v = u + at$, we make t the subject and find that:

$$t = \frac{v-u}{a} = -\frac{u}{a} = \frac{20}{7 \cdot 84} = 2 \cdot 55\,\text{s (3 s.f.)}.$$

3(a) We have equilibrium equations:

$$F_x = F\cos\theta - N\mu = ma, \ F_y = N + F\sin\theta - mg = 0$$

$$\Rightarrow F\cos\theta - mg\mu + F\mu\sin\theta = ma$$

$$\Rightarrow a = \frac{F(\cos\theta + \mu\sin\theta) - mg\mu}{m}.$$

(b) In the plane of the surface, the maximum force that may be applied without movement is $F = N\mu_s = (mg - F\sin\theta)\mu$. Equate this with the component $F\cos\theta$ of the applied force in the opposing direction to get:

$$F\cos\theta = mg\mu - (F\sin\theta)\mu$$

$$\Rightarrow F(\cos\theta + \mu\sin\theta) = mg\mu$$

$$\therefore F_{max} = \frac{mg\mu}{\cos\theta + \mu\sin\theta}.$$

4(a) We have the equations of motion of M and m respectively:

$$T - \mu N = T - \mu Mg = Ma$$

$$mg - T = ma;$$

substituting $T = mg - ma$ into the first equation gives:

$$mg - \mu Mg = Ma + ma = a(M + m)$$

$$\therefore a = \frac{g(m - \mu M)}{M + m};$$

masses will accelerate when $m - M\mu > 0$, otherwise the system remains at rest.

(b) The system is in motion but with 0 acceleration exactly when $m - M\mu = 0$, which is $\mu = \frac{m}{M} = \frac{1}{5}$.

5(a) We take the positive direction of motion to the right in the plane for M and downwards for m. The (frictional drag acting in the plane of the incline) is then $-\mu N = -\mu Mg\cos\theta$ and the component of gravity in the plane of the incline is $-Mg\sin\theta$. Hence our more general equations for the motions or M and of m become:

$$T - \mu Mg\cos\theta - Mg\sin\theta = Ma$$

$$mg - T = ma;$$

$$\Rightarrow mg - ma - \mu Mg\cos\theta - Mg\sin\theta = Ma$$

$$\therefore a = g\frac{m - M(\sin\theta + \mu\cos\theta)}{M + m}.$$

(b) On the other hand, if the mass M is slipping down the slope, then the frictional force reverses direction. If we again we measure the positive direction in the direction of the motion the equations take the form:

$$-T - \mu Mg\cos\theta + Mg\sin\theta = Ma$$

$$T - mg = ma$$

$$\Rightarrow -mg - ma - \mu Mg\cos\theta + Mg\sin\theta = Ma$$

$$\Rightarrow a = g\frac{M(\sin\theta - \mu\cos\theta) - m}{M + m}.$$

6. There are two possibilities. We may calculate a in each case. However if the result is $a < 0$, this contradicts our assumption, so that case is discarded. We first test Case (a):

$$a = 9\cdot 81\frac{2 - 4(\sin 45^\circ + 0\cdot 1\cos 45^\circ)}{4 + 2} = -1\cdot 8\,\mathrm{m/sec}^2 < 0;$$

so the acceleration is not upwards. Hence M must be sliding down the slope, as Case (b) confirms:

$$a = 9\cdot 81\frac{4(\sin 45^\circ - 0\cdot 1\cos 45^\circ) - 2}{4 + 2} = 0\cdot 9\,\mathrm{m/sec}^2.$$

7(a) For m_1 we have $2T - m_1 g = m_1 a_1$ while for m_2 we have $T - m_2 g = m_2 a_2$. Next, if m_2 moves a distance x downwards, then the portion of the string before the second pulley has decreased by x, so the portions to the left and right of the first pulley have each decreased by $\frac{x}{2}$. It follows that $a_2 = 2a_1$.

(b) We now solve as follows: $T = m_2 g + m_2 a_2 = m_2 g - 2m_2 a_1$. Hence we obtain:

$$2m_2 g - 4m_2 a_1 - m_1 g = m_1 a_1$$

$$\Rightarrow a_1(m_1 + 4m_2) = g(2m_2 - m_1)$$

$$\therefore a_1 = \frac{g(2m_2 - m_1)}{4m_2 + m_1}, \ a_2 = \frac{2g(2m_2 - m_1)}{4m_2 + m_1}$$

$$T = m_2 g - \frac{2m_2 g(2m_2 - m_1)}{4m_2 + m_1} = \frac{4m_2^2 g + m_1 m_2 g - 4m_2^2 g + 2m_1 m_2 g}{4m_2 + m_1}$$

$$\therefore T = \frac{3m_1 m_2 g}{4m_2 + m_1}.$$

Comment The main practical use of pulley systems is that they can provide *mechanical advantage*, meaning that it is possible for an object to be winced upwards at a constant speed (but zero acceleration) by a force that is only a fraction of the object's weight. This is possible by virtue of the work formula $W = Fd$ so that, for instance, the same work can be done (which may correspond to lifting a weight to a specific height against gravity) by a force of half that weight at the expense of moving twice the distance against the lesser force.

8(a) We have three equations:

$$m_1 g - T_1 = m_1 a_1, \ m_2 g - T_2 = m_2 a_2, \ m_3 g - T_2 = m_3 a_3.$$

Since the movable pulley has no (or at least negligible) mass we have $T_1 - 2T_2 = 0$, which is to say that $T_1 = 2T_2$. The acceleration of P_2 is $-a_1$. Let the acceleration of m_2 relative to P_2 be a. Then $a_2 = a - a_1$ and $a_3 = -a - a_1$, whence

$$a_1 + a_2 = a = -a_3 - a_1$$

$$\Rightarrow a_3 = -2a_1 - a_2.$$

(b)

$$\begin{bmatrix} m_1 & 0 & 2 \\ 0 & m_2 & 1 \\ -2m_3 & -m_3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ T_2 \end{bmatrix} = g \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}$$

9(a) The determinant D of the coefficient matrix is:

$$D = m_1(m_2 + m_3) + 2(0 - (-2m_2 m_3)) = m_1 m_2 + m_1 m_3 + 4m_2 m_3.$$

(b) Applying Cramer's rule to the system now gives:

$$a_1 = \frac{g \begin{vmatrix} m_1 & 0 & 2 \\ m_2 & m_2 & 1 \\ m_3 & -m_3 & 1 \end{vmatrix}}{D} = g \frac{m_1 m_2 + m_1 m_3 - 4m_2 m_3}{m_1 m_2 + m_1 m_3 + 4m_2 m_3};$$

$$a_2 = \frac{g\begin{vmatrix} m_1 & m_1 & 2 \\ 0 & m_2 & 1 \\ -2m_3 & m_3 & 1 \end{vmatrix}}{D} = g\frac{m_1m_2 - 3m_1m_3 + 4m_2m_3}{m_1m_2 + m_1m_3 + 4m_2m_3};$$

$$a_3 = -2a_1 - a_2 = g\frac{m_1m_3 - 3m_1m_2 + 4m_2m_3}{m_1m_2 + m_1m_3 + 4m_2m_3};$$

$$T_2 = \frac{g\begin{vmatrix} m_1 & 0 & m_1 \\ 0 & m_2 & m_2 \\ -2m_3 & -m_3 & m_3 \end{vmatrix}}{D} = \frac{4gm_1m_2m_3}{m_1m_2 + m_1m_3 + 4m_2m_3} = \frac{4g}{\frac{4}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}}.$$

$$T_1 = 2T_2 = \frac{8g}{\frac{4}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}}.$$

(c) If $m_1 = m_2 = m_3 = m$ then the above formulas simplify to:

$$a_1 = -\frac{g}{3},\ a_2 = a_3 = \frac{g}{3},\ T_1 = \frac{4gm}{3},\ T_2 = \frac{2gm}{3}.$$

10(a) We have

$$F - (m_1 + m_2 + m_3)\mu g = (m_1 + m_2 + m_3)a$$

$$\Rightarrow a = \frac{F - (m_1 + m_2 + m_3)\mu g}{m_1 + m_2 + m_3} = \frac{F}{m_1 + m_2 + m_3} - \mu g$$

(b) The tensions that arise in the equations of motion for m_2 and m_3 satisfy:

$$T_2 - T_1 - m_2\mu g = m_2 a;\ T_1 - m_1\mu g = m_1 a.$$

From the final equation and the answer to (a) we obtain:

$$T_1 = m_1(\mu g + \frac{F}{m_1 + m_2 + m_3} - \mu g) = \frac{m_1F}{m_1 + m_2 + m_3};$$

$$T_2 = T_1 + m_2\mu g + m_2(\frac{F}{m_1 + m_2 + m_3} - \mu g)$$

$$\Rightarrow T_2 = \frac{m_1F}{m_1 + m_2 + m_3} + \frac{m_2F}{m_1 + m_2 + m_3}$$

$$\therefore T_2 = \frac{(m_1 + m_2)F}{m_1 + m_2 + m_3}.$$

Problem Set 6

1. Let us move the point B upward by δy. The point A remains stationary. The virtual work done by the active force R_B is $R_B\delta y$. By similar triangles, the mass m moves upwards δ_m where $\frac{\delta_m}{a} = \frac{\delta y}{L}$ so that $\delta_m = \frac{a}{L}\delta y$. Hence the virtual work done by the weight is $-mg \cdot \frac{a}{L}\delta y$. By the Principle of virtual work we equate their sum to 0:

$$R_B\delta y - mg \cdot \frac{a}{L}\delta y = 0$$

$$\Rightarrow R_B = \frac{a}{L} \cdot mg$$

and, also as before, we obtain $R_A = (1 - \frac{a}{L})mg$.

2. The rotational work done by R_B is now $R_B L\delta\theta$ while that done by the weight is $-mg \cdot a\delta\theta$. Equating the sum of the work done by the torques of the active forces then gives:

$$R_B L\delta\theta = mga\delta\theta$$

$$\Rightarrow R_B = \frac{a}{L} \cdot mg$$

and similarly $R_A = (1 - \frac{a}{L})mg$.

3(a) Let y denote the vertical coordinate of the joint P where F acts and let $x = AB$. Then

$$y = L\cos\theta \Rightarrow \frac{dy}{d\theta} = -L\sin\theta$$

$$x = 2L\sin\theta \Rightarrow \frac{dx}{d\theta} = 2L\cos\theta.$$

(b) By $\overrightarrow{F}$, $\overrightarrow{\delta x}$ etc. we mean the vectors with the corresponding (non-negative) magnitudes F, δx, etc. in the direction of the force or displacement as the case may be. Consider the virtual work done when a small displacement $\delta\theta$ occurs in the angle θ, which may represent the single degree of freedom of the system. Then $\overrightarrow{F}$ acts in the direction of $\overrightarrow{\delta y}$ but $\overrightarrow{B_X}$ acts in the opposite direction to $\overrightarrow{\delta x}$. We may write the virtual work equation as:

$$\overrightarrow{F} \bullet \overrightarrow{\delta y} + \overrightarrow{B_X} \bullet \overrightarrow{\delta x} = 0.$$

Since the resistance force is in the opposite direction to the virtual displacement we obtain:

$$\Rightarrow FL\sin\theta\delta\theta - 2LB_X\cos\theta\delta\theta = 0$$

$$\therefore B_X = \frac{F}{2}\tan\theta.$$

4. The efficiency e is the ratio of the (unsigned magnitudes) of the output and input work done by the active forces.

$$e = \frac{B_X \delta x}{F \delta y} = \frac{\frac{F}{2} \tan\theta (2L \cos\theta \delta\theta)}{FL \sin\theta \delta\theta} = \tan\theta \cdot \cot\theta = 1.$$

5(a) By symmetry, the normal reaction force to F at both points A and B is $N = \frac{F}{2}$. Hence the frictional resistance force at the point B is $R = \frac{F\mu}{2}$ (in the negative x direction).

(b) The virtual work equation above, which was drawn up under the assumption that $\mu = 0$, now becomes:

$$\overrightarrow{F} \bullet \overrightarrow{\delta x} + \overrightarrow{(B_X + R)} \bullet \overrightarrow{\delta y} = 0$$

$$\Rightarrow FL \sin\theta \delta\theta - 2L(B_X + \frac{F\mu}{2}) \cos\theta \delta\theta = 0$$

$$\Rightarrow F \sin\theta - 2B_X \cos\theta - F\mu \cos\theta = 0$$

$$\therefore B_X = \frac{F}{2}(\tan\theta - \mu).$$

(c) Hence the efficiency ratio is given by:

$$\frac{B_X \delta x}{F \delta y} = \frac{\frac{F}{2}(\tan\theta - \mu) 2L \cos\theta \delta\theta}{FL \sin\theta \delta\theta} = (\tan\theta - \mu) \cot\theta = 1 - \mu \cot\theta.$$

(d) Since the efficiency cannot be negative we seek to interpret the inequality

$$1 - \mu \cot\theta \leq 0 \Leftrightarrow \cot\theta > \frac{1}{\mu}.$$

For fixed non-zero coefficient of friction μ this will be satisfied for all sufficiently small angles. For such small angles, no matter how large the force F, the frictional resistance will hold the strut in place, acting in effect like a second fixed point and so the notion of a virtual displacement in those circumstances is invalid. In other words our efficiency ratio calculation only applies if $\cot\theta \leq \mu^{-1}$.

6(a) The angle at B is also θ so that

$$\tan\theta = \frac{d}{y} \Rightarrow y = d \cot\theta.$$

By differentiation we find the virtual y-displacement as

$$\Rightarrow \frac{dy}{d\theta} = -\frac{d}{\sin^2\theta}.$$

(b)

$$\delta U = -P\delta y - M\delta\theta.$$

(c) Putting $\delta U = 0$ we obtain:

$$-\frac{Pd}{\sin^2\theta}\delta\theta - M\delta\theta = 0$$

$$\Rightarrow M = \frac{Pd}{\sin^2\theta}.$$

7(a) Again there is a single degree of freedom, represented either by the angle θ between the rods, or the coordinate x measuring the displacement from A to C. Taking the y direction to be positive in the downward direction, remembering that P lies at the midpoint of the rod BC, we have that

$$y_P = \frac{l}{2}\cos\frac{\theta}{2}$$

$$\Rightarrow \delta y_P = -\frac{l}{4}\sin\frac{\theta}{2}.$$

(b)

$$x_P = l\sin\frac{\theta}{2} + \frac{l}{2}\sin\frac{\theta}{2} = \frac{3}{2}l\sin\frac{\theta}{2}$$

$$\Rightarrow \delta x = \frac{3}{4}l\cos\frac{\theta}{2}\delta\theta.$$

(c)

$$\delta U = P\delta x + 2mg\delta y = 0$$

$$\Rightarrow P\cdot\frac{3}{4}l\cos\frac{\theta}{2}\delta\theta - 2mg\cdot\frac{l}{2}\sin\frac{\theta}{2}\delta\theta = 0$$

$$\Rightarrow \tan\frac{\theta}{2} = \frac{3P}{2mg}$$

$$\therefore \theta = 2\arctan\frac{3P}{2mg}.$$

8(a) We see that $y_C = 2l\sin\theta$ and $y_B = \frac{1}{2}y_C$.

(b) The elongation of the spring is $y_C - h = 2l\sin\theta - h$. Hence the spring force F is given by:

$$F = 2kl\sin\theta - kh. \tag{5}$$

(c) From (a)

$$\delta U = P\delta y_B - F\delta y_C = 0$$

$$\Rightarrow P\cdot\frac{1}{2}\delta y_C = F\delta_C$$

$$\therefore F = \frac{P}{2}.$$

Substituting in (5) now gives:

$$P = 4kl\sin\theta - 2kh$$

$$\sin\theta = \frac{P+2kh}{4kl}$$

$$\therefore \theta = \arcsin\Big(\frac{P+2kh}{4kl}\Big).$$

9. Let l denote the length of the supporting jack. Then

$$l^2 = b^2 + L^2 - 2bL\cos\theta \Rightarrow 2l\delta l = 2bL\sin\theta\delta\theta$$

$$h = 2b\sin\theta \Rightarrow \delta h = 2b\cos\theta\delta\theta.$$

The virtual work done by the two active forces (the weight and the jack) are given by:

$$-mg\delta h + F\delta l = 0$$

$$\Rightarrow F = \frac{mg\delta h}{\delta l} = \frac{mg\cdot 2b\cos\theta\delta\theta}{2bL\sin\theta\delta\theta/2l}$$

$$= \frac{2lmg\cot\theta}{L} = \frac{2mg\cot\theta}{L}\sqrt{b^2+L^2-2bL\cos\theta}$$

$$\therefore F = 2mg\cot\theta\sqrt{1+(\frac{b}{L})^2 - \frac{2b}{L}\cos\theta}.$$

10. Write P' for the image of each named point P under the rotation $\delta\theta$. Rotating about A, the angular displacements $r\delta\theta$ are in each case:

$$BB' = 5\delta\theta,\ CC' = 3\delta\theta,\ EE' = 6\delta\theta.$$

Applying $\delta U = 0$ where δU is the sum of the moments about A of the sum of the active forces we obtain, upon cancelling the common factor of $\delta\theta$, the equation for the vertical reaction R_B at B:

$$R_B\cdot BB' - (20\sin 60^\circ)CC' + (8\cos 45^\circ)EE' = 0$$

$$\Rightarrow 5R_B - \frac{20\sqrt{3}}{2}\cdot 3 + 4\sqrt{2}\times 6 = 0$$

$$\Rightarrow R_B = 6\sqrt{3} - \frac{24\sqrt{2}}{5} = 3\cdot 6041N \text{ (4d.p).}$$

Next, by Newton's law we have:

$$R_A + R_B = 20\sin 60^\circ - 8\sin 45^\circ$$

$$\Rightarrow R_A = (\frac{24\sqrt{2}}{5} - 6\sqrt{3}) + 10\sqrt{3} - 4\sqrt{2} = 4\sqrt{3} + \frac{4\sqrt{2}}{5} = 8\cdot 0596N \text{ (4 d.p.).}$$

Problem Set 7

1. We apply the chain rule in vector notation across each component. We have $\boldsymbol{F} = m\frac{d\boldsymbol{v}}{dt}$ and $d\boldsymbol{r} = \boldsymbol{v}dt$. Formally changing limits as we change the variable of integration gives:

$$W_{i,f} = \int_i^f \boldsymbol{F} \bullet d\boldsymbol{r} = \int_{i(t)}^{f(t)} m\frac{d\boldsymbol{v}}{dt} \bullet \boldsymbol{v}\, dt = m\int_{i(t)}^{f(t)} \boldsymbol{v}\bullet\frac{d\boldsymbol{v}}{dt}\, dt$$

$$= m\int_{i(\boldsymbol{v})}^{f(\boldsymbol{v})} \boldsymbol{v}\bullet d\boldsymbol{v} = \frac{1}{2}m(v_f^2 - v_i^2);$$

as, by Pythagoras, the sum of the squares of the component velocities equals the square of velocity of the particle.

2. Using the fact that the product rule is valid across vector products and that $\boldsymbol{p} = m\boldsymbol{v}$ we get:

$$\dot{\boldsymbol{L}} = \dot{\boldsymbol{r}} \times \boldsymbol{p} + \boldsymbol{r} \times \dot{\boldsymbol{p}} = \dot{\boldsymbol{r}} \times m\boldsymbol{v} + \boldsymbol{r} \times \dot{\boldsymbol{p}}$$

$$= m\dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}} + \boldsymbol{r}\times \boldsymbol{F} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{D}.$$

Comment In words, the rate of change of angular momentum equals the torque acting on the body. The law of the *conservation of angular momentum* is then that $\dot{\boldsymbol{L}} = \boldsymbol{0}$ when there is no torque acting.

3(a)

$$\boldsymbol{\tau} = \boldsymbol{r}\times m\boldsymbol{g} = l\hat{\boldsymbol{r}} \times mg(\cos\theta\hat{\boldsymbol{r}} - \sin\theta\hat{\boldsymbol{\theta}}) = -mgl(\sin\theta)\boldsymbol{k}.$$

(b) We equate the magnitude of the torque vector to $I\alpha = I\ddot{\theta}$. Now $I = ml^2$ so we obtain:

$$-mgl\sin\theta = ml^2\ddot{\theta}$$

$$\therefore \ddot{\theta} + \frac{g}{l}\sin\theta = 0.$$

(c) Since $\dot{r} = 0$, the transverse component of the equation $\boldsymbol{F} = m\boldsymbol{a}$ gives:

$$l\ddot{\theta} = -g\sin\theta$$

$$\therefore \ddot{\theta} + \frac{g}{l}\sin\theta = 0.$$

4(a) From $x = r\cos\theta$ we get

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

$$\Rightarrow \ddot{x} = \ddot{r}\cos\theta - \dot{r}\dot{\theta}\sin\theta - \dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta$$

$$\therefore \ddot{x} = (\ddot{r} - r\dot{\theta}^2)\cos\theta - (r\ddot{\theta} + 2\dot{r}\dot{\theta})\sin\theta.$$

Putting $\theta = 0$ and $\theta = \frac{3\pi}{2}$ respectively then gives the radial and transverse components of velocity, namely, $\dot{r}$ and $r\dot{\theta}$, and for acceleration $\ddot{r} - r\dot{\theta}^2$ and $r\ddot{\theta} + 2\dot{r}\dot{\theta}$. This follows as when $\theta = 0$ the x- component of these derivatives is in the same direction as the radial component, and the same applies to the transverse component when $\theta = \frac{3\pi}{2}$.

(b)

$$\frac{1}{r}\frac{d}{dt}\big(r^2\dot{\theta}\big) = \frac{1}{r}\big(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\big) = 2\dot{r}\dot{\theta} + r\ddot{\theta}.$$

5(a) Approximating $\sin\theta$ by θ yields the second order linear differential equation of simple harmonic motion:

$$\ddot{\theta} + \frac{g}{l}\theta = 0,$$

the auxiliary equation of which is $\lambda^2 = -\frac{g}{l}$ so that $\lambda = \pm\sqrt{\frac{g}{l}}i$. Hence the general solution to the equation of motion is given by $r(t) = l$ and

$$\theta(t) = C_1 \cos\sqrt{\frac{g}{l}}t + C_2 \sin\sqrt{\frac{g}{l}}t.$$

(b) We are given the initial conditions that $\dot{\theta}(0) = 0$ and $\theta(0) = \theta_0$. The latter gives $\theta_0 = C_1$. Then we have:

$$\dot{\theta}(t) = -\theta_0\sqrt{\frac{g}{l}}\sin\sqrt{\frac{g}{l}}t + C_2\sqrt{\frac{g}{l}}\cos\sqrt{\frac{g}{l}}t.$$

Putting $t = 0$ in this latter equation then gives:

$$0 = C_2\sqrt{\frac{g}{l}}\cos(0),$$

whence $C_2 = 0$ and so our particular solution is:

$$\theta(t) = \theta_0 \cos\sqrt{\frac{g}{l}}t,\ t \geq 0.$$

(c) The period T of our solution is then:

$$T = \frac{2\pi}{\sqrt{\frac{g}{l}}} = \frac{2\pi}{\sqrt{g}}\sqrt{l}.$$

6.

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r}\times(m\boldsymbol{\omega} \times \boldsymbol{r})$$
$$= m\big((\boldsymbol{r} \bullet \boldsymbol{r})\,\boldsymbol{\omega} - (\boldsymbol{r} \bullet \boldsymbol{\omega})\boldsymbol{r} = mr^2\,\boldsymbol{\omega} = I\omega\boldsymbol{k}.$$

7(a) Let $v = l\omega$ denote the magnitude of the velocity of the bob as it swings. Equating the loss of kinetic energy as the bob swings from the bottom of the circle, with initial velocity $v_0 = l\omega_0$, to the gain in potential energy we obtain the equation:

$$\frac{1}{2}mv_0^2 - \frac{1}{2}mv^2 = mgl(1-\cos\theta)$$
$$\Rightarrow v^2 = v_0^2 - 2gl(1-\cos\theta).$$

By Newton's law we have the tension T in the string satisfies:

$$T - mg\cos\theta = \frac{mv^2}{l}$$

$$\Rightarrow T = \frac{mv^2}{l} + mg\cos\theta = \frac{mv_0^2}{l} - 2mg(1-\cos\theta) + mg\cos\theta$$

$$\Rightarrow T = \frac{mv_0^2}{l} + mg(3\cos\theta - 2) \geq 0 \ \forall\theta$$
$$\Leftrightarrow v_0^2 \geq gl(2 - 3\cos\theta) \ \forall\theta;$$

the maximum value of the right hand side is $5gl$ (when $\theta = 180°$) so the minimum value of v_0 that will keep the string taut is $\sqrt{5gl}$. Since $l\omega_0 = v_0$ we see that the minimum angular velocity at the bottom of the swing is therefore

$$\omega_0 = \sqrt{\frac{5g}{l}}.$$

7(b) Unlike the string, the rod may support a negative tension, so $T < 0$ is allowed. However, the system has to have enough energy for the bob to reach the top of the circle. Hence we require that $v \geq 0$ for all angles, which, since the maximum of $1 - \cos\theta$ is 2, we require that

$$v^2 = v_0^2 - 2lg(1-\cos\theta) \geq 0$$

$$\Leftrightarrow v_0^2 \geq 4lg$$

so that $v_0 > \sqrt{4lg}$ and

$$\omega_0 > \sqrt{\frac{4g}{l}}.$$

We do require strict inequality as equality here would see the bob stopping right at the top in *(unstable) equilibrium.*

8(a) Conservation of energy here takes the form $\frac{1}{2}I\dot{\theta}^2 = mga\sin\theta$, so that, since $I = \frac{4}{3}ma^2$ (as the length of the rod is $2a$) we obtain:

$$\frac{1}{2}\cdot\frac{4}{3}ma^2\dot{\theta}^2 = mga\sin\theta$$

$$\Rightarrow a\dot{\theta}^2 = \frac{3}{2}g\sin\theta$$

$$\Rightarrow 2a\dot{\theta}\ddot{\theta} = \frac{3}{2}g(\cos\theta)\dot{\theta}$$

$$\therefore a\ddot{\theta} = \frac{3}{4}g\cos\theta.$$

(b) Let X and Y denote the respective forces applied by the hinge to the rod in the directions BA and perpendicular to the rod at A. Then since the centre of mass lies a distance a from A and $a\dot{\theta}^2$ and $a\ddot{\theta}$ are the components of acceleration in the directions of X and of Y respectively we have from Newton's law:

$$X - mg\sin\theta = ma\dot{\theta}^2$$

$$-Y + mg\cos\theta = ma\ddot{\theta}$$

whence

$$X = mg\sin\theta + ma\dot{\theta}^2 = \frac{5}{2}mg\sin\theta$$

$$Y = mg\cos\theta - ma\ddot{\theta} = \frac{1}{4}mg\cos\theta.$$

9. Let F and R denote the friction and the reactive forces respectively and let θ be the displacement angle of the rod from the vertical at a time before slipping occurs. Denote the length of the rod by $2a$. Then Conservation of energy gives:

$$\frac{1}{2}\cdot\frac{4}{3}ma^2\dot{\theta}^2 + mga\cos\theta = mga$$

$$\Rightarrow a\dot{\theta}^2 = \frac{3}{2}g(1-\cos\theta)$$

$$\Rightarrow a\ddot{\theta} = \frac{3}{4}g\sin\theta.$$

Hence as in the previous question, we have by Newton's law applied to the centre of mass of the toppling rod that:

$$F = ma\ddot{\theta}\cos\theta - ma\dot{\theta}^2\sin\theta; \tag{6}$$

$$R - mg = -ma\ddot{\theta}\sin\theta - ma\dot{\theta}^2\cos\theta \tag{7}$$

$$(6) \Rightarrow F = \frac{3}{4}mg\sin\theta\cos\theta - \frac{3}{2}mg(1-\cos\theta)\sin\theta$$

$$\Rightarrow F = \frac{3}{4}mg\sin\theta(3\cos\theta - 2)$$

$$(7) \Rightarrow R = mg - \frac{3}{4}mg\sin^2\theta - \frac{3}{2}mg(1-\cos\theta)\cos\theta$$

$$= \frac{1}{4}mg(4 - 3\sin^2\theta - 6\cos\theta + 6\cos^2\theta)$$

$$\Rightarrow R = \frac{1}{4}mg(1-3\cos\theta)^2.$$

At the slipping point when $\theta = \frac{\pi}{6}$, we have $F = R\mu$ so that

$$\mu = \frac{F}{R} = \frac{\frac{3}{4}\sin\frac{\pi}{6}(3\cos\frac{\pi}{6} - 2)}{\frac{1}{4}(1 - 3\cos\frac{\pi}{6})^2} = \frac{\frac{3}{8}(\frac{3\sqrt{3}}{2} - 2)}{\frac{1}{4}(1 - \frac{3\sqrt{3}}{2})^2} = \frac{3(\frac{3\sqrt{3}-4}{2})}{(\frac{2-3\sqrt{3}}{2})^2}$$

$$= \frac{6(3\sqrt{3} - 4)}{31 - 12\sqrt{3}} = 0 \cdot 3513.$$

10. Let the door be swinging with angular velocity ω_0 when it hits the stop and rebound with angular velocity ω_1. Upon collision there will be a rebound impulse P perpendicular to the door and an impulse J at the hinge A also perpendicular to the door. Again let the width of the door AB be $2a$ and let x denote the distance of the stop from A. The moment imparted by the impulse equals the change in angular momentum of the door. Taking the direction of rebound as positive rotation, and taking the change in angular momentum about A we may express that as an equation:

$$Px = \frac{4}{3}ma^2\omega_1 - (-\frac{4}{3}ma^2\omega_0) = \frac{4}{3}ma^2(\omega_1 + \omega_0)$$

$$\Rightarrow \omega_1 = \frac{3Px}{4ma^2} - \omega_0.$$

The velocities of the centre of mass of the door before and after impact are respectively $-a\omega_0$ and $a\omega_1$. We now equating the total moment of impulse about the centre of mass with the change in angular momentum. We have P acting in the positive sense along with ω_1 while ω_0 acts in the negative direction of rotation. The perpendicular direction of J (clockwise or anti-clockwise about the centre of mass, which corresponds to up or down at A) is not clear; indeed the question is asking for when $J = 0$. In order to be definite we write J in the positive sense. This gives:

$$J + P = ma\omega_1 - (-ma\omega_0) = ma(\frac{3Px}{4ma^2} - \omega_0 + \omega_0)$$

$$\Rightarrow J = ma(\frac{3Px}{4ma^2}) - P = P(\frac{3x}{4a} - 1).$$

It follows that $J = 0$ exactly when $1 = \frac{3x}{4a}$, which is to say that $x = \frac{4}{3}a$. Since the width of the door was $2a$, this means that the stop should be placed at $\frac{2}{3}$ the width of the door, measured from A, to avoid any jarring at the hinge when the door strikes the stop.

Comment This problem is often represented as finding the *centre of percussion* or *sweet spot* on a baseball or cricket bat (treating the bat as a simple rod). When the ball strikes the sweet spot, the batter feels no jarring of the hands and maximum energy is imparted to the ball. Note that the impulse J may be either positive or negative depending on whether or not $x > \frac{4}{3}a$ or $x < \frac{4}{3}a$. A right-handed batter will feel the bat pushed into their left hand if the impact is near the toe end of the bat but will feel the bat pushed back into their right hand if the impact is close to the bat handle.

Problem Set 8

1(a) In general the number of constraints is $3N - s$ where N is the number of particles and s the number of *holomonic constraints*. Here $N = 1$ and $s = 2$, the two constraints being $z = 0$ and x and y are linked by the equation

$$\frac{y}{x} = \tan(\omega t).$$

Hence there is $3 \times 1 - 2 = 1$ degree of freedom.

(b) Here we have $V = 0$ so that

$$L = T = \frac{1}{2}mv^2 = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$$

where m is the mass of the sphere.

(c) We have

$$x = r\cos\omega t,\ \ y = r\sin\omega t$$

$$\Rightarrow \dot{x} = \dot{r}\cos\omega t - r\omega\sin\omega t,\ \ \dot{y} = \dot{r}\sin\omega t + r\omega\cos\omega t$$

$$\Rightarrow v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\omega^2$$

$$\therefore L(r, \dot{r}, t) = \frac{m}{2}(\dot{r}^2 + r^2\omega^2).$$

2. We now solve

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{r}}) - \frac{\partial L}{\partial r} = 0 \tag{8}$$

In our case we have

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{r}}) = \frac{d}{dt}(m\dot{r}) = m\ddot{r},\ \ \frac{\dot{\partial} L}{\partial r} = m\omega^2 r,$$

so that, upon cancelling m, (8) becomes:

$$\ddot{r} - \omega^2 r = 0.$$

Putting $r(t) = e^{At}$ we obtain the auxiliary equation $A^2 - \omega^2 = 0$ so our general solution to the problem is:

$$r(t) = C_1 e^{\omega t} + C_2 e^{-\omega t}.$$

Comment As $t \to \infty$ the solution is dominated by the term $C_1 e^{\omega t}$, indicating that the sphere will eventually by flung out of the tube by the centrifugal force.

3(a)

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m((a\dot{\theta}(1-\cos\theta))^2 + (-a\dot{\theta}\sin\theta)^2)$$

$$\therefore T = ma^2\dot{\theta}^2(1-\cos\theta)$$

Since $V = mgy = mga(1+\cos\theta)$ we get:

$$L = T - V = ma^2\dot{\theta}^2(1-\cos\theta) - mga(1+\cos\theta).$$

(b)

$$\frac{\partial L}{\partial \theta} = ma^2\dot{\theta}^2\sin\theta + mga\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = 2ma^2\dot{\theta}(1-\cos\theta)$$

$$\Rightarrow \frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) = 2ma^2(\ddot{\theta}(1-\cos\theta) + \dot{\theta}^2\sin\theta).$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) - \frac{\partial L}{\partial \theta} = 2ma^2(\ddot{\theta}(1-\cos\theta) + \dot{\theta}^2\sin\theta)) - ma^2\dot{\theta}^2\sin\theta - mga\sin\theta = 0$$

$$\therefore \ddot{\theta}(1-\cos\theta) + \frac{1}{2}\dot{\theta}^2\sin\theta - \frac{g}{2a}\sin\theta = 0 \qquad (9)$$

4. Dividing (9) throughout by $1-\cos\theta$ gives:

$$\ddot{\theta} + \frac{\dot{\theta}^2\sin\theta}{1-\cos\theta} - \frac{g}{2a}\frac{\sin\theta}{1-\cos\theta} = 0.$$

Now

$$\frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos\frac{\theta}{2}} = \cot\frac{\theta}{2} \text{ and so we obtain for (9):}$$

$$\ddot{\theta} + \dot{\theta}^2\cot\frac{\theta}{2} - \frac{g}{2a}\cot\frac{\theta}{2} = 0 \qquad (10)$$

5. To solve (10) we put $u = \cos\frac{\theta}{2}$ so that $\dot{u} = -\frac{1}{2}\dot{\theta}\sin\frac{\theta}{2}$ and $\ddot{u} = -\frac{1}{2}\ddot{\theta}\sin\frac{\theta}{2} - \frac{1}{2}\dot{\theta}^2\cos\frac{\theta}{2}$. We then multiply (10) throughout by $\sin\frac{\theta}{2}$ to give:

$$\ddot{\theta}\sin\frac{\theta}{2} + \dot{\theta}^2\cos\frac{\theta}{2} - \frac{g}{2a}\cos\frac{\theta}{2} = 0$$

$$\Rightarrow \ddot{u} + \frac{g}{4a}u = 0.$$

The auxiliary equation to this differential equation is $r^2 + \frac{g}{4a} = 0$ and since $g, a > 0$ we have roots $r = \pm i\sqrt{\frac{g}{4a}}$. Hence we recover our solution:

$$u(t) = \cos(\frac{\theta}{2}) = C_1\cos\sqrt{\frac{g}{4a}}t + C_2\sin\sqrt{\frac{g}{4a}}t.$$

6(a) Resolving forces tangentially we have:

$$-mg\sin\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = ml\ddot{\theta}$$

$$\therefore \ddot{\theta} + \frac{g}{l}\sin\theta = 0.$$

(b) We may take θ, the angle between the pendulum line and the vertical, as the generalized coordinate for the one degree of freedom of the problem. We then get:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(l\dot{\theta})^2 = \frac{1}{2}ml^2\dot{\theta}^2;$$

$$y = l - l\cos\theta \Rightarrow V = mgl(1-\cos\theta)$$

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$$

$$\Rightarrow \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta},\ \frac{d}{dt}(\frac{\partial L}{\partial\dot{\theta}}) = ml^2\ddot{\theta},$$

$$\frac{\partial L}{\partial\theta} = -mgl\sin\theta;$$

$$\Rightarrow ml^2\ddot{\theta} - mgl\sin\theta = 0$$

$$\therefore \ddot{\theta} + \frac{g}{l}\sin\theta = 0.$$

Comment Often, for simple problems such as this one, the scheme based on Newton's laws is simpler than the Euler-Lagrange equation based on energy. However, the latter method is strategically simple and so is more easily translated into software for the purpose of computation.

7(a) We have the constraints $x^2 + y^2 + z^2 = R^2$ for the coordinates of the mass and also $\phi = \omega t$, where ϕ is the angle the plane of the hoop makes with the x-axis. Since there is 1 particle and 2 constraints there is $3 \times 1 - 2 = 1$ degree of freedom. We take as our generalized coordinate the angle θ between the z-axis and the radius of the hoop to the bead.

(b) Projecting the radius of the hoop on to the xy-plane we see that:

$$x = R\sin\theta\cos\omega t,\ y = R\sin\theta\sin\omega t,\ z = R\cos\theta$$

$$\Rightarrow \dot{x} = R\dot{\theta}\cos\theta\cos\omega t - R\omega\sin\theta\sin\omega t,\ \dot{y} = R\dot{\theta}\cos\theta\sin\omega t + R\omega\sin\theta\cos\omega t,\ \dot{z} = -R\dot{\theta}\sin\theta.$$

(c)

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2),\ V = mgz$$

$$\Rightarrow L = T - V = \frac{1}{2}mR^2(\dot{\theta}^2\cos^2\theta\cos^2\omega t + \dot{\theta}^2\cos^2\theta\sin^2\omega t + \dot{\theta}^2\sin^2\theta)$$

$$+\frac{1}{2}R^2\omega^2(\sin^2\theta\sin^2\omega t + \sin^2\theta\cos^2\omega t) - mgR\cos\theta;$$

$$\therefore L = \frac{1}{2}mR^2(\dot{\theta}^2 + \omega^2\sin^2\theta) - mgR\cos\theta.$$

8.

$$\frac{\partial L}{\partial \dot\theta} = mR^2\dot\theta,\ \frac{\partial L}{\partial \theta} = mR^2\omega^2 \sin\theta\cos\theta + mgR\sin\theta,\ \frac{d}{dt}(\frac{\partial L}{\partial \dot\theta}) = mR^2\ddot\theta;$$

$$\Rightarrow mR^2\ddot\theta = mR^2\omega^2\sin\theta\cos\theta + mgR\sin\theta$$

$$\therefore \ddot\theta = \frac{1}{2}\omega^2\sin 2\theta + \frac{g}{R}\sin\theta.$$

9(a) There are 4 constraints, which may be expressed as $z_m = 0 = z_M$, $y_M = 0$, $\frac{h-y_m}{x_m-x_M} = \tan\alpha$. There are 2 particles, so that there are $3\times 2 - 4 = 2$ degrees of freedom. The coordinates (x_m, y_m) determine the position of m and of the wedge, using the constraints. Now

$$x_m = x_M + q\cos\alpha,\ y_m = h - q\sin\alpha$$

$$\Rightarrow \dot x_m = \dot x_m + \dot q\cos\alpha,\ \dot y_m = -\dot q\sin\alpha.$$

so we may take q and x_M as our generalized coodinates.

(b)

$$T = \frac{1}{2}m(\dot x_m^2 + \dot y_m^2) + \frac{1}{2}M\dot x_M^2,\ V = mgy_m$$

$$\Rightarrow L = T - V = \frac{1}{2}m(\dot x_M^2 + \dot q^2 + +2\dot x_M\dot q\cos\alpha) + \frac{1}{2}M\dot x_M^2 - mg(h - q\sin\alpha).$$

10(a) Write down the Euler-Lagrange equation for each generalized coordinate:

$$\frac{d}{dt}(m\dot x_M + m\dot q\cos\alpha + M\dot x_M) = 0$$

$$\Rightarrow (m+M)\ddot x_M + m\ddot q\cos\alpha = 0 \tag{11}$$

$$\frac{d}{dt}(m\dot q + m\dot x_M\cos\alpha) = mg\sin\alpha$$

$$\Rightarrow \ddot q + \ddot x_M\cos\alpha = g\sin\alpha. \tag{12}$$

(b) Substituting from (12) into (11) now gives:

$$(m+M)\ddot x_M + m(g\sin\alpha - \ddot x_M\cos\alpha)\cos\alpha = 0$$

$$\Rightarrow \ddot x_M(M + m - m\cos^2\alpha) = -mg\sin\alpha$$

$$\Rightarrow \ddot x_M = -\frac{mg\sin\alpha}{M + m\sin^2\alpha};$$

$$\ddot q = g\sin\alpha + \frac{mg\sin\alpha}{M + m\sin^2\alpha}.$$

Finally

$$\ddot x_m = \ddot q + x_M = g\sin\alpha.$$

Comment We note that both accelerations are constant and the acceleration of m is independent of M and m. If m is small compared to M then $\ddot x_M$ is small and $\ddot{q}$ approaches the acceleration we would see with a fixed wedge. However, if m is very large compared to M and α is small then both $\ddot{x}_M$ and $\ddot{q}$ become arbitrarily large in magnitude as the wedge is fired out from underneath the heavy weight pressing down on it.

Problem Set 9

1(a) $L = mrv_t$, where v_t is the tangential component of the velocity of the mass. Hence

$$L = mr(r\omega) = mr^2\omega.$$

(b) From $\tau = \dot{L}$ we obtain:

$$\tau = 2m\omega r\frac{dr}{dt} = rF_c$$

$$\therefore F_c = 2m\omega\frac{dr}{dt} = 2m\omega v_r.$$

2(a)

$$v_B = v_A + \omega r.$$

(b) The centrifugal acceleration on the circumference is

$$\frac{v_B^2}{r} = \frac{(v_A + \omega r)^2}{r} = \frac{v_A^2}{r} + 2\omega v_A + \omega^2 r.$$

3. By inspection we see that we require the standard rotation matrix:

$$A = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}.$$

4.

$$\begin{aligned} \boldsymbol{a} &= a'_x\boldsymbol{i}' + a'_y\boldsymbol{j}' + a'_z\boldsymbol{k}' \\ &= a'_x(\boldsymbol{i}\cos\phi + \boldsymbol{j}\sin\phi) + a'_y(-\boldsymbol{i}\sin\phi + \boldsymbol{j}\cos\phi) + a'_z\boldsymbol{k} \\ &= (a'_x\cos\phi - a'_y\sin\phi)\boldsymbol{i} + (a'_x\sin\phi + a'_y\cos\phi)\boldsymbol{j} + a'_z\boldsymbol{k}. \end{aligned}$$

5.

$$\begin{aligned} \frac{d\boldsymbol{a}}{dt} &= \dot{a}_x\boldsymbol{i} + \dot{a}_y\boldsymbol{j} + \dot{a}_z\boldsymbol{k} \\ &= \overline{(a'_x\cos\phi - a'_y\sin\phi)}^{\boldsymbol{\cdot}}\boldsymbol{i} + \overline{(a'_x\sin\phi + a'_y\cos\phi)}^{\boldsymbol{\cdot}}\boldsymbol{j} + a'_z\boldsymbol{k} \\ &= (\dot{a_x}'\cos\phi - a'_x\sin\phi\dot{\phi} - \dot{a}'_y\sin\phi - a'_y\cos\phi\dot{\phi})\boldsymbol{i} + \\ &\quad (\dot{a}'_x\sin\phi + a'_x\cos\phi\dot{\phi} + \dot{a}'_y\cos\phi - a'_y\sin\phi\dot{\phi})\boldsymbol{j} + \dot{a'_z}\boldsymbol{k} \\ &= \dot{a}'_x(\cos\phi\boldsymbol{i} + \sin\phi\boldsymbol{j}) + \dot{a}'_y(-\sin\phi\boldsymbol{i} + \cos\phi\boldsymbol{j}) + \dot{a}'_z\boldsymbol{k} + \end{aligned}$$

$$a'_x\dot{\phi}(-\sin\phi\boldsymbol{i}+\cos\phi\boldsymbol{j})+a'_y\dot{\phi}(-\cos\phi\boldsymbol{i}-\sin\phi\boldsymbol{j})$$
$$=\dot{a_x}'\boldsymbol{i}'+\dot{a_y}'\boldsymbol{j}'+\dot{a_z}'\boldsymbol{k}'+\dot{\phi}a'_x\boldsymbol{j}'-\dot{\phi}a_y\boldsymbol{i}'$$
$$=\frac{D\boldsymbol{a}}{Dt}+\dot{\phi}(a'_x\boldsymbol{j}'-a'_y\boldsymbol{i}').$$

6. Now

$$\boldsymbol{\Omega}\times\boldsymbol{a}=\dot{\phi}\boldsymbol{k}\times(a'_x\boldsymbol{i}'+a'_y\boldsymbol{j}'+a'_z\boldsymbol{k}')=\dot{\phi}(a'_x\boldsymbol{j}'-a'_y\boldsymbol{i}')$$
$$\therefore\frac{d\boldsymbol{a}}{dt}=\frac{D\boldsymbol{a}}{Dt}+\boldsymbol{\Omega}\times\boldsymbol{a}. \tag{13}$$

7(a) Putting $\boldsymbol{a}=\boldsymbol{r}$ in (13) we obtain:

$$\frac{d^2\boldsymbol{r}}{at^2}=\frac{d}{dt}(\frac{D\boldsymbol{r}}{dt}+\boldsymbol{\Omega}\times\boldsymbol{r})=\frac{D}{Dt}(\frac{D\boldsymbol{r}}{dt}+\boldsymbol{\Omega}\times\boldsymbol{r})+\boldsymbol{\Omega}\times(\frac{D\boldsymbol{r}}{dt}+\boldsymbol{\Omega}\times\boldsymbol{r})$$
$$=\frac{D^2\boldsymbol{r}}{Dt^2}+\frac{D\boldsymbol{\Omega}}{Dt}\times\boldsymbol{r}+\boldsymbol{\Omega}\times\frac{D\boldsymbol{r}}{dt}+\boldsymbol{\Omega}\times\frac{D\boldsymbol{r}}{dt}+\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})$$
$$=\frac{D^2\boldsymbol{r}}{Dt^2}+\frac{D\boldsymbol{\Omega}}{Dt}\times\boldsymbol{r}+2\boldsymbol{\Omega}\times\frac{D\boldsymbol{r}}{dt}+\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}).$$

(b) Since in the inertial frame S we have $\boldsymbol{F}=m\frac{d^2\boldsymbol{r}}{dt^2}$ it follows from (a) that

$$m\frac{D^2\boldsymbol{r}}{Dt^2}=\boldsymbol{F}-m\frac{D\boldsymbol{\Omega}}{Dt}\times\boldsymbol{r}-2m\boldsymbol{\Omega}\times\frac{D\boldsymbol{r}}{dt}-m\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}).$$

Comment The equation of motion of the particle in the rotating frame involves three components, the second of which is the *Coriolis force* and the third is related to the *centrifugal force*. If, as is the case with a rotating planet, we have that $\dot{\phi}$ is constant then $\frac{D\boldsymbol{\Omega}}{Dt}=\boldsymbol{0}$ and the previous equation simplifies accordingly (as in the next question).

8(a) In the given scenario we have $\frac{D\boldsymbol{r}}{Dt}=\frac{D^2\boldsymbol{r}}{Dt^2}=\boldsymbol{0}=\frac{D\boldsymbol{\Omega}}{Dt}$, the latter because the Earth rotates with constant angular velocity. The external force $\boldsymbol{F}=-m\boldsymbol{g}$ so we obtain:

$$-m\boldsymbol{g}-m\,\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})=-m\boldsymbol{g}_e.$$

(b) The vector $\boldsymbol{\Omega}=\boldsymbol{k}\dot{\phi}$ is in the direction of axis of rotation. Hence $\boldsymbol{\Omega}\times\boldsymbol{r}$ is directed due East (the direction of rotation of the Earth) so that $\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})$ is directed towards the axis of rotation. Since this is substracted from $\boldsymbol{g}$ to get $\boldsymbol{g}_e$, the *effective gravitational acceleration*, it follows that $g_e<g$ and g_e is least at the equator.

9.

$$\boldsymbol{\Omega}\times\boldsymbol{r}=\Omega R\sin(\frac{\pi}{2}-\lambda)\hat{\boldsymbol{n}}$$

where $\hat{\boldsymbol{n}}$ is the unit vector directed due East. Hence

$$\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})=\Omega^2R\cos\lambda\hat{\boldsymbol{n}}'$$

where $\hat{\boldsymbol{n}}'$ is the unit vector directed towards the axis of rotation. Since $\boldsymbol{g}_e = \boldsymbol{g} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega} \times \boldsymbol{r})$ we obtain from the vector triangle together with the Cosine rule that:

$$g_e^2 = g^2 + \Omega^4 R^2 \cos^2 \lambda - 2g\Omega^2 R \cos^2 \lambda$$
$$\therefore g_e^2 = g^2 + \Omega^2 R \cos^2 \lambda(\Omega^2 R - 2g).$$

10. The minimum value for g_e occurs at the equator for there we hae $\lambda = 0$ and so $\cos\lambda = 1$ and the previous equation becomes:

$$g_e^2 = g^2 + \Omega^4 R^2 - 2g\Omega^2 R = (g - \Omega^2 R)^2$$
$$\Rightarrow g_e = g - (\Omega R)^2.$$

Now $R = 6 \cdot 371 \times 10^6$m and 24 hours equals $24 \times 60 \times 60 = 8 \cdot 64 \times 10^4$s. Hence

$$\Omega = \frac{2\pi}{8.64 \times 10^4} = 7.272 \times 10^{-5}$$
$$\Rightarrow g_e = g - (7 \cdot 272^2 \times 6 \cdot 371) \times (10^{-10} \times 10^6)$$
$$= g - 3.369 \times 10^{-2}$$
$$\Rightarrow \frac{g_e}{g} = 1 - 0 \cdot 003434 = 0 \cdot 9966.$$

Which is to say that equatorial effective weight is 99.66% that of the poles.

Problem Set 10

1(a) We begin with $\boldsymbol{a} \bullet \boldsymbol{a} = |\boldsymbol{a}|^2 = a^2$. Differentiating with respect to time:

$$\boldsymbol{a}\bullet\frac{d\boldsymbol{a}}{dt} + \frac{d\boldsymbol{a}}{\boldsymbol{dt}}\bullet \boldsymbol{a} = 2|\boldsymbol{a}|\frac{d|\boldsymbol{a}|}{dt}$$
$$\therefore \boldsymbol{a} \bullet \frac{d\boldsymbol{a}}{dt} = |\boldsymbol{a}|\frac{d|\boldsymbol{a}|}{dt}.$$

(b) Taking $\boldsymbol{a} = \dot{\boldsymbol{r}} = \boldsymbol{v}$ in part (a) we obtain:

$$m\int \ddot{\boldsymbol{r}} \bullet \dot{\boldsymbol{r}}\, dt = m\int |\boldsymbol{v}|\frac{d|\boldsymbol{v}|}{dt}\, dt = m\int |\boldsymbol{v}|\, d|\boldsymbol{v}| = \frac{1}{2}m|\boldsymbol{v}|^2 + c.$$

2(a) From $m\ddot{\boldsymbol{r}} = F(r)\hat{\boldsymbol{r}}$ we take the dot product with $\dot{\boldsymbol{r}}$ and use Question 1(b) and 1(a) in turn to find:

$$\frac{1}{2}m|\boldsymbol{v}|^2 = \int F(r)\hat{\boldsymbol{r}} \bullet \frac{d\boldsymbol{r}}{dt}\, dt = \int \frac{F(r)}{r}\boldsymbol{r} \bullet \frac{d\boldsymbol{r}}{dt}\, dt = \int \frac{F(r)}{r} r\frac{dr}{dt}\, dt$$
$$= \int F(r)\, dr;$$

hence the energy equation has the form:

$$\frac{1}{2}m|\boldsymbol{v}|^2 - \int F(r)\,dr = E,$$

the value of the arbitrary constant E depending on the zero level set for the potential energy $-\int F(r)\,dr$ of the system in question.

(b) Putting $F(r) = r^{-2}$ we get $\int F(r)\,dr = \int \frac{dr}{r^2} = -\frac{1}{r} + c$ so that $V(r) = \frac{1}{r} + c$.

3(a) We have $\dot{x} = a$, $\ddot{x} = 0$;$\dot{y} = 2bt$, $\ddot{y} = 2b$, $\dot{z} = 3ct^2$, $\ddot{z} = 6ct$. Hence

$$\boldsymbol{F} = m\,\ddot{\boldsymbol{r}} = m(0,\,2b,\,6ct).$$

(b)

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ at & bt^2 & ct^3 \\ 0 & 2bm & 6cmt \end{vmatrix}$$

$$= m(4bct^3,\,-6act^2,\,2abt).$$

(c)

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = m(\boldsymbol{r} \times \boldsymbol{v}) = m\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ at & bt^2 & ct^3 \\ a & 2bt & 3ct^2 \end{vmatrix}$$

$$= m(bct^4,\,-2act^3,\,abt^2).$$

(d)

$$\dot{\boldsymbol{L}} = m(4bct^3,\,-6act^2,\,2abt) = \boldsymbol{M}.$$

4(a)

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{r} \times \frac{F(r)}{r}\,\boldsymbol{r} = \frac{F(r)}{r}(\boldsymbol{r} \times \boldsymbol{r}) = \frac{F(r)}{r}\boldsymbol{0} = \boldsymbol{0}.$$

(b) We have in general that $\dot{\boldsymbol{L}} = \boldsymbol{M}$ and since $\boldsymbol{M} = \boldsymbol{0}$, it follows that $\boldsymbol{L}$ is constant.

5(a) Since $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$, where $\boldsymbol{p} = m\dot{\boldsymbol{r}}$ is the linear momentum of P we get

$$\boldsymbol{r} \bullet \boldsymbol{L} = \boldsymbol{r} \bullet (\boldsymbol{r} \times \boldsymbol{p}) = \boldsymbol{0}$$

as any triple product (scalar or vector) with a repeated vector is zero. Since $\boldsymbol{L}$ is constant, $\boldsymbol{r} \bullet \boldsymbol{L} = \boldsymbol{0}$ is the equation of the plane through the origin with normal vector $\boldsymbol{L}$.

(b) The area δA swept out in time δt by the radial vector $\boldsymbol{r} = OP$ satisfies

$$\delta A = |\frac{1}{2}\,\boldsymbol{r} \times \delta\boldsymbol{r}|$$

$$\Rightarrow \frac{dA}{dt} = \lim_{\delta t \to 0} \frac{\delta A}{\delta t} = \lim_{\delta t \to 0} |\frac{1}{2}\,\boldsymbol{r} \times \frac{\delta \boldsymbol{r}}{\delta t}|$$

$$= \frac{1}{2}|\,\boldsymbol{r} \times \dot{\boldsymbol{r}}| = \frac{|\boldsymbol{L}|}{2m};$$

since $\boldsymbol{L}$ and m are each constants, it follows that $\frac{dA}{dt}$ is also constant. This in turn implies that the change in area swept out by the radial vector is the same for any two time intervals of equal duration, which is *Kepler's second law* when applied to the inverse square law of planetary motion.

6(a) Since $\boldsymbol{L} = m\boldsymbol{r} \times \dot{\boldsymbol{r}}$ it follows that

$$\dot{\boldsymbol{r}} \times \dot{\boldsymbol{L}} = \dot{\boldsymbol{r}} \times \big((m\dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}}) + (m\boldsymbol{r} \times \ddot{\boldsymbol{r}})\big) = m\dot{\boldsymbol{r}} \times (\boldsymbol{r} \times \ddot{\boldsymbol{r}}) = \boldsymbol{0}$$

as $\ddot{\boldsymbol{r}}$ and $\boldsymbol{r}$ are assumed to be parallel vectors. Hence

$$\frac{d}{dt}(m\dot{\boldsymbol{r}} \times \boldsymbol{L}) = m\ddot{\boldsymbol{r}} \times \boldsymbol{L} + m\dot{\boldsymbol{r}} \times \dot{\boldsymbol{L}}$$

$$\Rightarrow m\frac{d}{dt}(\ddot{\boldsymbol{r}} \times \boldsymbol{L}) = m\ddot{\boldsymbol{r}} \times \boldsymbol{L} + \boldsymbol{0}$$

$$\Rightarrow \frac{d}{dt}(\ddot{\boldsymbol{r}} \times \boldsymbol{L}) = m\ddot{\boldsymbol{r}} \times (\boldsymbol{r} \times \dot{\boldsymbol{r}}) = \frac{F(r)}{r}\boldsymbol{r} \times (\boldsymbol{r} \times \dot{\boldsymbol{r}})$$

$$\Rightarrow \frac{d}{dt}(\dot{\boldsymbol{r}} \times \boldsymbol{L}) = \frac{F(r)}{r}\big((\boldsymbol{r} \bullet \dot{\boldsymbol{r}})\boldsymbol{r} - (\boldsymbol{r} \bullet \boldsymbol{r})\dot{\boldsymbol{r}}\big)$$

(b) We have $\boldsymbol{r} \bullet \boldsymbol{r} = r^2$, and differentiating this gives $2\,\boldsymbol{r} \bullet \dot{\boldsymbol{r}} = 2r\dot{r}$ so that $\boldsymbol{r} \bullet \dot{\boldsymbol{r}} = r\dot{r}$. Continuing we obtain:

$$\frac{d}{dt}(\dot{\boldsymbol{r}} \times \boldsymbol{L}) = \frac{F(r)}{r}(r\dot{r}\boldsymbol{r} - r^2\dot{\boldsymbol{r}}) = F(r)\dot{r}\boldsymbol{r} - F(r)r\dot{\boldsymbol{r}}.$$

7. We equate

$$\frac{d}{dt}(\alpha(r)\boldsymbol{r}) = \frac{d\alpha}{dt}\boldsymbol{r} + \alpha\frac{d\boldsymbol{r}}{dt} = F(r)\dot{r}\boldsymbol{r} - F(r)r\dot{\boldsymbol{r}}$$

$$\Rightarrow \frac{d\alpha}{dt} = F(r)\frac{dr}{dt} \;\&\; \alpha = -F(r)r$$

$$\Rightarrow \frac{d\alpha}{dt} = -\frac{\alpha}{r}\frac{dr}{dt}$$

$$\Rightarrow r\frac{d\alpha}{dt} + \alpha\frac{dr}{dt} = \frac{d}{dt}(r\alpha) = 0$$

$$\Rightarrow \alpha = \frac{\lambda}{r}$$

where λ is an arbitrary constant. From the equation $\alpha = -F(r)r$ we finally conclude that

$$F(r) = -\frac{\alpha}{r} = -\frac{\lambda}{r^2}.$$

8. By Question 7, for an inverse square law, we have

$$\frac{d}{dt}(\dot{\boldsymbol{r}} \times \boldsymbol{L}) = \frac{d}{dt}(\alpha(r)\boldsymbol{r}) = \frac{d}{dt}(\lambda\frac{\boldsymbol{r}}{r})$$

$$\Rightarrow \dot{\boldsymbol{r}} \times \boldsymbol{L} - \lambda \frac{\boldsymbol{r}}{r} = \boldsymbol{K}$$

where $\boldsymbol{K}$ is then a constant vector.

Comment The vector $\boldsymbol{K}$ (often denoted as a lower case $\boldsymbol{k}$) is known as the *Lenz-Runge vector*.

(b) Since $\boldsymbol{L} = m\boldsymbol{r} \times \dot{\boldsymbol{r}}$ and $\boldsymbol{L}$ is constant, it follows that the particle moves in a plane π through the origin determined by the fixed vector $\boldsymbol{L}$ acting as a normal to π, and each of $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ are perpendicular to $\boldsymbol{L}$. Since each of $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ lie in π, it follows that $\boldsymbol{K}$ also lies in π and $\boldsymbol{K}$ is perpendicular to $\boldsymbol{L}$ as

$$\boldsymbol{K} \bullet \boldsymbol{L} = (\dot{\boldsymbol{r}} \times \boldsymbol{L}) \bullet \boldsymbol{L} - \frac{\lambda}{r}(\boldsymbol{r} \bullet \boldsymbol{L}) = 0 - 0 = 0,$$

as $\dot{\boldsymbol{r}} \times \boldsymbol{L}$ is perpendicular to $\boldsymbol{L}$ and $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ is perpendicular to $\boldsymbol{r}$.

We conclude that, if non-zero, the trio of vectors $(\boldsymbol{L},\boldsymbol{K},\boldsymbol{L} \times \boldsymbol{K})$ form an orthogonal basis at each point P of the path of the particle.

9(a) We have

$$\ddot{r} + \frac{\mu}{m} r = 0,$$

which has solution

$$\boldsymbol{r} = \boldsymbol{a} \sin \omega t + \boldsymbol{b} \cos \omega t$$

for arbitrary constant vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and where $\omega = \sqrt{\frac{\mu}{m}}$.

(b)

$$\dot{\boldsymbol{r}} = \boldsymbol{a}\omega \cos \omega t - \boldsymbol{b}\omega \sin \omega t$$

$$\begin{aligned} \boldsymbol{L} = m\,\boldsymbol{r}\times \dot{\boldsymbol{r}} &= m(\sin \omega t \boldsymbol{a} + \cos \omega t \boldsymbol{b}) \times (\omega \cos \omega t \boldsymbol{a} - \omega \sin \omega t \boldsymbol{b}) \\ &= m(\omega \sin^2 \omega t + \omega \cos^2 \omega t)(\boldsymbol{b} \times \boldsymbol{a}) \\ \therefore \boldsymbol{L} &= m\omega(\boldsymbol{b} \times \boldsymbol{a}). \end{aligned}$$

(c)

$$E = \frac{1}{2} m |\dot{\boldsymbol{r}}|^2 - \int F(r)\, dr.$$

$$\frac{1}{2} m |\dot{\boldsymbol{r}}|^2 = \frac{1}{2} m\omega^2 (|\boldsymbol{a}|^2 \cos^2 \omega t + |\boldsymbol{b}|^2 \sin^2 \omega t - 2\boldsymbol{a} \bullet \boldsymbol{b} \sin \omega t \cos \omega t) \tag{14}$$

$$\int F(r)\, dr = -\mu \int r\, dr = -\frac{m\omega^2}{2} r^2$$

$$= -\frac{1}{2} m\omega^2 (|\boldsymbol{a}|^2 \sin^2 \omega t + |\boldsymbol{b}|^2 \cos^2 \omega t + 2\boldsymbol{a} \bullet \boldsymbol{b} \sin \omega t \cos \omega t) \tag{15}$$

Taking the required difference of (14) and (15) gives:

$$E = \frac{1}{2} m\omega^2 (|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2) = \frac{\mu}{2}(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2).$$

10.

$$\dot{\boldsymbol{L}} = m\ddot{\boldsymbol{r}} \times \boldsymbol{r} = -\frac{m\lambda}{r^3}(\boldsymbol{r} \times \boldsymbol{r}) = \boldsymbol{0}. \tag{16}$$

$$\dot{\boldsymbol{K}} = \ddot{\boldsymbol{r}} \times \boldsymbol{L} + \dot{\boldsymbol{r}} \times \dot{\boldsymbol{L}} - \lambda \frac{\dot{\boldsymbol{r}} r - \dot{r} \boldsymbol{r}}{r^2} \tag{17}$$

Now by (16), $\dot{\boldsymbol{L}} = \boldsymbol{0}$ and as for the first term in (17) we find:

$$\ddot{\boldsymbol{r}} \times \boldsymbol{L} = m\ddot{\boldsymbol{r}} \times (\boldsymbol{r} \times \dot{\boldsymbol{r}}) = -\frac{\lambda \boldsymbol{r}}{r^3} \times (\boldsymbol{r} \times \dot{\boldsymbol{r}})$$

$$= \frac{\lambda}{r^3}(\boldsymbol{r} \times (\dot{\boldsymbol{r}} \times \boldsymbol{r})) = \frac{\lambda}{r^3}\big((\boldsymbol{r} \bullet \boldsymbol{r})\dot{\boldsymbol{r}} - (\boldsymbol{r} \bullet \dot{\boldsymbol{r}})\boldsymbol{r}\big)$$

and since $\boldsymbol{r} \bullet \boldsymbol{r} = r^2$ implies that $\boldsymbol{r} \bullet \dot{\boldsymbol{r}} = r\dot{r}$ we obtain:

$$= \frac{\lambda}{r^3}\big(r^2 \dot{\boldsymbol{r}} - r\dot{r}\boldsymbol{r}\big) = \frac{\lambda}{r^2}(r\dot{\boldsymbol{r}} - \dot{r}\boldsymbol{r})$$

and so we obtain that the terms in (17) do indeed cancel to give **0**.