MA101 Algebra & Complex Numbers Solutions

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Solutions and Comments for the Problems

Problem Set 1

1.

$$\frac{1}{3\sqrt{5}-\sqrt{7}} = \frac{1}{3\sqrt{5}-\sqrt{7}} \cdot \frac{3\sqrt{5}+\sqrt{7}}{3\sqrt{5}+\sqrt{7}} = \frac{3\sqrt{5}+\sqrt{7}}{9\cdot 5-7} = \frac{3\sqrt{5}+\sqrt{7}}{38}$$

.

Comment: This is a fundamental algebraic technique, which we see mirrored in division of complex numbers $a+ib$ through use of the complex conjugate $a-bi$.

2.

$$1-x^2+3x = -(x^2-3x-1) = -((x-\frac{3}{2})^2 - \frac{9}{4} - 1) = -(x-\frac{3}{2})^2 + \frac{13}{4}.$$

Hence $a=-1, b=\frac{3}{2}, c=\frac{13}{4}$.

Comment The point to note is that when given a quadratic expression ax^2+bx+c it is best to factorize it by 'brute force' as $a(x^2+\frac{b}{a}x+\frac{c}{a})$ and then go on to deal with quadratic expression inside the brackets where the coefficient of x is unity (i.e. the number 1).

3. $y = x^2-6x+13 = (x-3)^2+4$. Hence $a=3, b=4$.

4. We need to maximize the expression $x-x^2$, which is the same as minimizing its negative $y=x^2-x=(x-\frac{1}{2})^2-\frac{1}{4}$. Since the minimum of a square occurs by setting it equal to 0, the minimum value occurs at $x=\frac{1}{2}$ and the value at that minimum is evidently $\frac{1}{4}$. Therefore $\frac{1}{2}$ is the number that most exceeds its own square.

Comment Most students solve this problem using calculus: if $y = x - x^2$ then $y' = 1 - 2x$ and putting this derivative to 0 gives the answer. Another argument is that $x - x^2 = x(1-x)$; the roots of this quadratic expression are then 0 and 1 so the turning point is, by symmetry of the parabola, half way in between. The lesson to learn is that completing the square will allow you to answer nearly any question about a given quadratic expression, while inspection of factorizations of algebraic expressions likewise often reveals useful information.

5. Writing the equation as $c^{-1}(a+b) = c^{-1}a + c^{-1}b$ we see that this is a consequence of the *Distributive Law* (of multiplication over addition), that is $x(y+z) = xy + xz$.

Comment The Distributive Law is a key Law of Algebra as it is the only one that links the two fundamental operations of addition and multiplication. Of course interchanging the operations of addition and multiplication to get a Distributive Law of Multiplication over Addition would give: $x + yz = (x+y)(x+z)$, which is never true except in very special circumstances (which are easily determined). This contrasts with the behaviour of sets under the operations of intersection and union, $\cap$ and $\cup$, where both forms of the Distribution Law hold: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

6. $S = 2\pi rh + 2\pi r^2 \Rightarrow h = \frac{S - 2\pi r^2}{2\pi r}$. Thus $V = \frac{\pi r^2}{2\pi r}\big(S - 2\pi r^2\big) =$

$$\frac{r}{2}\big(S - 2\pi r^2\big) = \frac{1}{2}rS - \pi r^3.$$

7. We have $\pi r^2 h = x^3$, where x denotes the length of the cube's edge. Hence $x = (\pi r^2 h)^{\frac{1}{3}} = (\pi \cdot 8^2 \cdot 14)^{\frac{1}{3}} = 4(14\pi)^{\frac{1}{3}} = 141$mm, to the nearest mm.

8.

$$g(x) - f(x) = 3x^3 + 5x^2 + 5x + 2 - x^2 - 5x - 2 = x^2(3x+4),$$

which is negative if and only if $3x + 4 < 0$, that is if and only if $x < -\frac{4}{3}$.

9. By the Remainder Theorem, we substitute $x = 1$ into the polynomial (as 1 is the root of $x - 1 = 0$) to obtain the answer $10 - 8 + 6 + 4 - 2 + 5 = 15$.

10. By the Factor Theorem, we put $x = -2$ (the root of $x + 2 = 0$) and equate the outcome to 0 to get:

$$-8 - 4k - 6 + 7k = 0 \Leftrightarrow 3k = 14 \Leftrightarrow k = \frac{14}{3}.$$

Problem Set 2

1.

$$1 - 4x \leq -\frac{1}{2}x - 1 \Leftrightarrow 2 \leq \frac{7}{2}x \Leftrightarrow x \geq \frac{4}{7}.$$

The *least* value of x satisfying this inequality is $x = \frac{4}{7}$.

Comment An important practical point for a student to bear in mind is to answer the question asked. Many students stop after finding that $x \geq \frac{4}{7}$.

2.

$$-4 \leq \frac{4-2x}{3} < 4 \Rightarrow -12 \leq 4 - 2x < 12 \Rightarrow -16 \leq -2x < 8 \Rightarrow 8 \geq x > 4 :$$

that is $-4 < x \leq 8$.

3.

$$\begin{aligned} &4x - 2 < x + 8 < 9x + 1 \\ &\Rightarrow 3x < 10 \,\&\, 7 < 8x \\ &\Rightarrow \frac{7}{8} < x < \frac{10}{3}. \end{aligned}$$

4. We need to locate the intersections of boundaries: $x + y = 3$ meets the $x = 0$ line at $(0, 3)$ and $y = 6$ meets the $x = 0$ axis at $(0, 6)$ and meets the line $y = 4x - 5$ at $(\frac{11}{4}, 6)$. Finally we solve $x + y = 3$ and $y = 4x - 5$; substituting $y = 3 - x$ into the latter gives $3 - x = 4x - 5 \Rightarrow 5x = 8 \Rightarrow x = \frac{8}{5}$; hence $y = 3 - \frac{8}{5} = \frac{7}{5}$ and so gives the corner point $(\frac{8}{5}, \frac{7}{5})$. The four points $(0, 3), (0, 6), (\frac{11}{4}, 6)$ and $(\frac{8}{5}, \frac{7}{5})$ are the vertices of the quadrilateral; the horizontal line $y = 6$ is hatched to indicate that the inequality $y < 6$ is strict.

5.

$$\begin{aligned} &\frac{x+3}{x-1} \geq -2 \Rightarrow \frac{(x+3) + 2(x-1)}{x-1} \geq 0 \Rightarrow \frac{3x+1}{x-1} \geq 0 \\ &\Rightarrow (3x+1)(x-1) \geq 0. \end{aligned}$$

The quadratic expression represents an upward opening parabola with roots $-\frac{1}{3}$ and 1. Hence the solution set is:

$$(-\infty, -\frac{1}{3}] \cup (1, \infty).$$

Comment: We note that the original inequality forces us to exclude $x = 1$. In general we need to take care when multiplying an inequality by an expression the sign of which depends on a variable involved, if necessary breaking the ensuing argument into cases depending on sign. This is circumvented here by multiplying by $(x-1)^2 > 0$ for $x \neq 1$.

6.
$$\frac{x-1}{1-3x} > 7 \Rightarrow \frac{x-1}{3x-1} + 7 < 0 \Rightarrow \frac{(x-1)+7(3x-1)}{3x-1} < 0$$
$$\Rightarrow \frac{22x-8}{3x-1} < 0 \Rightarrow \frac{11x-4}{3x-1} < 0 \Rightarrow (11x-4)(3x-1) < 0$$
$$\Rightarrow \frac{1}{3} < x < \frac{4}{11}.$$

7(a)
$$|2x+1| \le 5 \Leftrightarrow -5 \le 2x+1 \le 5 \Leftrightarrow -6 \le 2x \le 4$$
$$\Leftrightarrow -3 \le x \le 2.$$

Comment Or we may look at it as follows:

$$|2x+1| \le 5 \Leftrightarrow |2\big(x+\frac{1}{2}\big)| \le 5 \Leftrightarrow |2| \cdot |x+\frac{1}{2}| \le 5 \Leftrightarrow |x-(-\frac{1}{2})| \le \frac{5}{2},$$

which is an interval of half-length $\frac{5}{2}$, centred at $x = -\frac{1}{2}$. Hence $|2x+1| \le 5 \Leftrightarrow -\frac{1}{2}-\frac{5}{2} \le x \le -\frac{1}{2}+\frac{5}{2} \Leftrightarrow -3 \le x \le 2$. The former method certainly is quicker and clearer in this instance but the alternative geometric approach is useful when we move into the complex plane where such an inequality leads to a circle with the 'half-length' then representing the circle's radius.

(b)
$$|1-4x| = |4x-1| > 9$$
$$\Leftrightarrow (4x-1>9)\,\text{or}\,(4x-1<-9)$$
$$\Leftrightarrow (x<-2)\,\text{or}\,(x>\frac{5}{2}).$$

8(a) We see that $-1+(-3) < x+y < 5+-1$, that is

$$-4 < x+y < 4.$$

8(b) $|x+2| \le 5$ and $|-2y-2| < 8$. The first inequality can be written as $-5 \le x+2 \le 5$, which is $-7 \le x \le 3$. As for the second we have

$$|-2(y+1)| = |-2||y+1| < 8 \Rightarrow |y+1| < 4 \Rightarrow -4 < y+1 < 4$$
$$\Rightarrow -5 < y < 3.$$

We need to identify the range of values of xy as (x,y) ranges over this rectangular domain. The upper limit of xy is 35 approached as $(x,y) \to (-7,-5)$, which is not attained because of the strict inequality $-5 < y$. The lower limit of xy is -21 approached as $(x,y) \to (-7,3)$, again not obtained. Hence the full range of xy is given by:
$$-21 < xy < 35.$$

9. We note that the equation is equivalent to $ax+b = \pm(cx+d)$ so we solve the pair of linear equations

$$(ax + b = cx + d)\ (ax + b = -cx - d)$$

$$\Leftrightarrow (a - c)x = d - b\ (a + c)x = -b - d$$

$$\Leftrightarrow x = \frac{d-b}{a-c} \text{ or } x = -\frac{b+d}{a+c}.$$

If $a = c$ the first equation however has no solution, (unless $b = d$ in which case the equation becomes $ax + b = ax + b$ which is trivially true for all x) and the second possibility represents the unique solution to our original equation. A similar remark applies if $a = -c$ and then the first solution is the only one except if $b = -d$ in which case the second equation holds for all x.

Otherwise both values of x as given are solutions of the equation. These solutions would coincide if and only if

$$\frac{b-d}{a-c} = \frac{b+d}{a+c} \Leftrightarrow (a + c)(b - d) = (a - c)(b + d)$$

$$\Leftrightarrow ab - ad + bc - cd = ab + ad - bc - cd \Leftrightarrow -ad + bc = ad - bc$$

$$2(ad - bc) = 0 \Leftrightarrow ad = bc.$$

In conclusion, the equation has two distinct solutions unless either $ad = bc$, or $a = \pm c$, in which case the solution is unique.

Comment Anticipating matrices and determinants, we note that the condition that $ad \neq bc$ can be written as $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$. In terms of the graphs of the functions $y = |ax+b|$ and $y = |cx+d|$ we observe that they each form an infinite V with the point of the V on the x-axis. We then see that when comparing the two graphs, the inner ray of the V that has the shallower gradient will meet both rays of the steeper V, giving two solutions. However, if $ad = bc$, then the graphs share a common corner, while if $a = \pm c$ or the corresponding rays of the respective V's are parallel and the inner pair of rays intersect.

We also since both sides of our equation are non-negative an equivalent quadratic equation is $(ax+b)^2 = (cx+d)^2$, which can be solved as the difference of two squares.

10. If sign(x) =sign(y) (1st and 3rd quadrants) then $\frac{1}{x} \leq \frac{1}{y} \Leftrightarrow y \leq x$. If $x < 0, y > 0$ (2nd quadrant) then $\frac{1}{x} < 0 < \frac{1}{y}$, while if $x > 0, y < 0$ (4th quadrant) then $\frac{1}{x} > 0 > \frac{1}{y}$. Hence the region in which $\frac{1}{x} \leq \frac{1}{y}$ consists of the 2nd quadrant and those areas in the 1st and 3rd quadrants that lie on or below the line $y = x$; the cartesian axes themselves are not included.

Problem Set 3

1. We are given $2^{x^2+5x} = \frac{1}{16} = 2^{-4}$. Equating exponents yields:

$$x^2 + 5x = -4 \Leftrightarrow (x+1)(x+4) = 0 \Leftrightarrow x \in \{-1, -4\}.$$

2. Annual rate is $x\%$ and x satisfies:

$$4,000\big(1 + \frac{x}{100}\big)^5 = 6,740.23$$

$$\Rightarrow 1 + \frac{x}{100} = \Big(\frac{6,740.23}{4,000}\Big)^{\frac{1}{5}} = 1.11;$$

and so $x\% = 11\%$.

3.

$$5,000 = 10,000(1 - \frac{x}{100})^3 \Rightarrow 1 - \frac{x}{100} = 2^{-\frac{1}{3}}$$

$$\Rightarrow \frac{x}{100} = 1 - 2^{-\frac{1}{3}} \Rightarrow x = 20.6\%.$$

4. Subtracting 1 from both sides of the given equation yields:

$$\log_2(\log_2 x) = -1 \Leftrightarrow \log_2 x = 2^{-1} = \frac{1}{2}$$

$$\Leftrightarrow x = 2^{\frac{1}{2}} = \sqrt{2}.$$

5. Putting $y = \log_2 x$ gives us:

$$y^2 + 3y - 10 = (y-2)(y+5) = 0$$

$$\Rightarrow \log_2 x \in \{2, -5\} \Rightarrow x \in \{2^2, 2^{-5}\}.$$

Hence $x = 4$ or $x = \frac{1}{32}$.

6. We have upon putting $y = \log_3 x$ the equation

$$y - \frac{2}{y} = 1 \Rightarrow y^2 - y - 2 = (y+1)(y-2) = 1.$$

Hence either $y = -1$ so that $\log_3 x = -1$ and $x = 3^{-1}$, or $y = 2$ and $\log_3 x = 2$ and so $x = 3^2 = 9$. Therefore $x \in \{\frac{1}{3}, 9\}$.

7. We have:

$$\log_a(4x) - 3\log_a(x^2) - \log_a(128) = \log_a \frac{4x}{2^7 x^6} = 0$$

$$\Rightarrow \log_a(2^{-5}x^{-5}) = \log_a((2x)^{-5}) = -5\log_a(2x) = 0 \Rightarrow 2x = 1$$
$$\Rightarrow x = \frac{1}{2}.$$

8. We have

$$\log_3(\log_8 x) = -1 \Rightarrow \log_8 x = 3^{-1} \Rightarrow x = 8^{\frac{1}{3}} = 2.$$

.

9. Here we have

$$9^{2\log_3 2} = 3^{2(\log_3 4)} = 3^{\log_3 16} = 16.$$

10. Solutions will arise when the exponent equals 0, so we put $q(x) = x^2 - 11x + 30 = (x-5)(x-6) = 0$, giving the two values $x = 5, 6$. Writing $p(x) = x^2 - 5x + 5$ we note that $p(5) = 5 \neq 0$ and $p(6) = 11 \neq 0$ so these two solutions to $q(x) = 0$ are solutions to the given equation. More solutions will arise when $p(x) = \pm 1$ so we solve

$$p(x) = x^2 - 5x + 5 = 1 \Rightarrow x^2 - 5x + 4 = (x-1)(x-4) = 0,$$

giving $x = 1, 4$. Additional solutions potentially arise when $p(x) = -1$ but for no other values of $p(x)$. Hence we solve $p(x) = -1$:

$$x^2 - 5x + 6 = (x-2)(x-3) = 0$$

giving $x = 2, 3$. Now $q(2) = 12$ and $q(3) = 6$ are both even, so that these two values do indeed yield further solutions. The solution set to our original equation is therefore $\{1, 2, 3, 4, 5, 6\}$.

Problem Set 4

1. $2x + y + 1 = 0 \Rightarrow y = -2x - 1$. Hence the slope of the required line is $m = -(-2)^{-1} = \frac{1}{2}$. Next use the given point to find the intercept c: $2 = \frac{1}{2}(1) + c \Rightarrow c = \frac{3}{2}$. Therefore $y = \frac{1}{2}x + \frac{3}{2}$.

2.
$$1 = |x| + |-2x| = |x| + |-2| \cdot |x| = |x| + 2|x| = 3|x|.$$
Hence $|x| = \frac{1}{3}$ so that the two solutions are $x = \pm\frac{1}{3}$.

3.
$$\sqrt{3-x} = x - 3 \Rightarrow 3 - x = (x-3)^2 = (3-x)^2$$
$$\Leftrightarrow (x = 3) \text{ or } (3 - x = 1)$$

and so $x = 3$ or $x = 2$; however, $x = 2$ is not a solution of the original equation. Hence the solution is just $x = 3$.

4.

$$\sqrt{2+x} + x = 10 \Leftrightarrow \sqrt{2+x} = 10 - x \Rightarrow 2 + x = (10-x)^2 = 100 - 20x + x^2$$
$$\Leftrightarrow x^2 - 21x + 98 = 0 \Leftrightarrow (x-14)(x-7) = 0$$
$$\Leftrightarrow x = 7 \text{ or } x = 14;$$

but only $x = 7$ satisfies the original equation.

Comment Remember that when you carry out an operation that is not one-to-one (such as squaring) to both sides of an equation the new equation that results is not equivalent to the original but may have more solutions than the one you are seeking to solve. For that reason the solutions that result must be tested to see which ones, if any, solve the original equation. In terms of logical quantifiers, the connection between the equations is only in the forward direction and the step cannot in general be reversed. In this case we are using a manipulation of the form $A = B \Rightarrow A^2 = B^2$; in general $A^2 = B^2$ does not imply that $A = B$ so that a solution to the latter equation is not necessarily a solution to the former. However all solutions to $A = B$ lie among the solutions to $A^2 = B^2$. The squared equation is equivalent to $\pm\sqrt{2+x} + x = 10$.

5. $\sqrt{a^2+b^2} = a + b \Rightarrow a^2 + b^2 = (a+b)^2 \Rightarrow 2ab = 0$. Hence $a = 0$ or $b = 0$. If, say $b = 0$, then $\sqrt{a^2} = a \Rightarrow a \geq 0$; similarly if $a = 0$ then $b \geq 0$. We conclude that $\sqrt{a^2+b^2} = a+b$ if and only if at least one of a and b equals 0 <u>and</u> $a, b \geq 0$.

Comment Always be aware that $\sqrt{a^2} = |a|$.

6. Substitution yields three equations:

$$a + b + c = -6 \quad (1)$$
$$9a - 3b + c = 30 \quad (2)$$
$$c = -3 \quad (3)$$

Putting (3) into both(1) and (2) gives $a + b = -3$ and $9a - 3b = 33$. Hence we have

$$3a + 3b = -9$$
$$9a - 3b = 33;$$

and adding these equations gives $12a = 24 \Rightarrow a = 2$. Finally $b = -3 - a = -3 - 2 = -5$. Our equation is therefore:

$$y = 2x^2 - 5x - 3.$$

7. A point (x, y) is equidistant from the points $(-1, 2)$ and $(-5, -3)$ if and only if

$$(x-(-1))^2+(y-2)^2=(x-(-5))^2+(y-(-3))^2$$

as this says exactly that the squares of the separations, and hence the separations themselves, are equal. Hence we get:

$$(x+1)^2+(y-2)^2=(x+5)^2+(y+3)^2$$

$$\Rightarrow (x^2+2x+1)+(y^2-4y+4)=(x^2+10x+25)+(y^2+6y+9)$$

$$\Rightarrow 2x-4y+5=10x+6y+34 \Rightarrow 8x+10y+29=0 \text{ or}$$

$$y=-\frac{4}{5}x-\frac{29}{10}.$$

Alternatively we can argue geometrically that the answer is the perpendicular bisector B of the line L joining the given points. The slope of L is given by $\frac{(2-(-3))}{-1-(-5)}=\frac{5}{4}$. The midpoint of the segment L is the intersection of L and B and has co-ordinates:

$$P=(\frac{-1+(-5)}{2}, \frac{2+(-3)}{2})=(-3,-\frac{1}{2}).$$

The slope of B is the negative reciprocal of that of L and so B has the form:

$$y=-\frac{4}{5}x+c,$$

where c is determined by substitution of the co-ordinates of the common point P:

$$-\frac{1}{2}=-\frac{4}{5}(-3)+c \Rightarrow c=-\frac{1}{2}-\frac{12}{5}=-\frac{29}{10}$$

and therefore the equation of B is $y=-\frac{4}{5}x-\frac{29}{10}$.

8. From $y=e^{1-x}$ we get

$$1-x=\ln y \Rightarrow x=1-\ln y$$

and so, writing x as the independent variable we have that the inverse function is $y=1-\ln x$.

9. Standard examples are $f((x)=a-x$ and $f(x)=\frac{k}{x}$ $(k \neq 0)$. In the respective cases we get

$$f(f(x))=a-(a-x)=a-a+x=x;$$

$$f(f(x))=\frac{k}{(\frac{k}{x})}=k \cdot \frac{x}{k}=x.$$

10. The man drinks at a rate $m=\frac{1}{12}$ bottles/day. His friend drinks at an unknown rate of f bottles/day. However $m+f=\frac{1}{8}$ bottles/day and so

$f = \frac{1}{8} - \frac{1}{12} = \frac{3-2}{24} = \frac{1}{24}$ bottles/day. Therefore a bottle of whiskey would last his friend 24 days.

Comment Another version of this problem asks you to find out how long it would take to fill a bath with hot water given the time to do so for the cold tap and a combined time for both taps together. This particular calculation, which involves the reciprocal of the sum of reciprocals, also arises in the calculation of the effective electrical resistance R of two resistors, R_1 and R_2, connected in parallel: $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.

Problem Set 5

1. The roots r_1, r_2, and r_3 of the polynomial satisfy the equation and so:

$$x^3 - 4x^2 + 10x - \frac{19}{2} = 0 = (x - r_1)(x - r_2)(x - r_3).$$

Equating coefficients of x^2 gives:

$$-4 = -r_1 - r_2 - r_3 \Rightarrow r_1 + r_2 + r_3 = 4.$$

2. $\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)$. Now

$$(x - \alpha)(x - \beta) = x^2 + 3x + 4 = 0 \Rightarrow \alpha\beta = 4,\ \alpha + \beta = -3;$$

Now since α and β solve our equation we have $\alpha^2 = -3\alpha - 4,\ \beta^2 = -3\beta - 4$

$$\Rightarrow \alpha^2 + \beta^2 = -3(\alpha + \beta) - 8 = -3(-3) - 8 = 9 - 8 = 1.$$

This all serves to show that

$$\alpha^3 + \beta^3 = (\alpha + \beta)((\alpha^2 + \beta^2) - \alpha\beta) = (-3)(1 - 4) = (-3)(-3) = 9.$$

Alternatively,

$$(\alpha + \beta)^3 = \alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta)$$
$$\Rightarrow \alpha^3 + \beta^3 = (-3)^3 - 3(4)(-3) = -27 + 36 = 9.$$

3.

$$x^2 + 2x + 3 = (x - \alpha)(x - \beta) \Rightarrow \alpha\beta = 3,\ \alpha + \beta = -2.$$

Write

$$x^2 + bx + c = (x - (1 - \frac{1}{\alpha}))(x - (1 - \frac{1}{\beta})).\text{ Then}$$

$$c = (1 - \frac{1}{\alpha})(1 - \frac{1}{\beta}) = \frac{(\alpha - 1)(\beta - 1)}{\alpha\beta} = \frac{\alpha\beta - (\alpha + \beta) + 1}{\alpha\beta}$$

$$= \frac{3-(-2)+1}{6} = \frac{6}{3} = 2.$$

$$\text{Next } b = -(1-\tfrac{1}{\alpha}) - (1-\tfrac{1}{\beta}) = -2 + \tfrac{1}{\alpha} + \tfrac{1}{\beta} = -2 + \tfrac{\alpha+\beta}{\alpha\beta}$$

$$= -2 + \frac{(-2)}{3} = -\frac{8}{3}.$$

Hence a required equation is:

$$x^2 - \frac{8}{3}x + 2 = 0 \Leftrightarrow 3x^2 - 8x + 6 = 0.$$

4. The *Associative Law.*

5. By division, $\frac{2x^2+x-4}{x-1} = 2x+3-\frac{1}{x-1}$. An asymptote arises at the point of singularity $x = 1$. As $|x| \to \infty$, $f(x) \to 2x+3$ so the asymptotes of the curve are: $x = 1$, and $y = 2x+3$.

6. $2x^4 + x^2 - 6 = (2x^2-3)(x^2+2) = 0$ so that $x = \pm\sqrt{2}i$ or $x = \pm\frac{\sqrt{6}}{2}$.

7. Working through the factors of 12, we find -2 is a root, so we divide by the factor $x-(-2) = x+2$. The result of the division by $x+2$ is x^2+3x-6, a polynomial with roots $\frac{1}{2}(-3 \pm \sqrt{9+24}) = \frac{-3\pm\sqrt{33}}{2}$. Hence the full factorization can be expressed as:

$$x^3 + 5x^2 - 12 = (x+2)(x + \frac{3+\sqrt{33}}{2})(x + \frac{3-\sqrt{33}}{2}).$$

Comment Any rational root of a polynomial with integer coefficients can be found by use of the *Rational Root Theorem*: if $\frac{p}{q}$ is a rational root of $a_0 + a_1x + \cdots + a_nx^n$ cancelled to lowest terms then p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n. This gives a finite list of possible candidates for the rational roots of the polynomial so all such roots can then be found by trial. In particular, any cubic with a rational root can always be factored into linear terms as in this example. The proof of the Rational Root Theorem is simple and just amounts to looking at what happens when the root $\frac{p}{q}$ is substituted into the polynomial and then invoking elementary observations of factors of sums to gain the stated conclusions.

8. We have $y = 4(x-1)$ and so $x^2 + 16(x^2-2x+1) = 17x^2 - 32x + 15 = 0$. Hence

$$x = \frac{32 \pm \sqrt{32^2 - 60\times 17}}{34} = \frac{32 \pm \sqrt{4}}{34} = 1 \text{ or } \tfrac{15}{17}.$$

For $x = 1$ we have $y = 4(1-1) = 0$; for $x = \frac{15}{17}$ we have $y = 4(\frac{15}{17}-1) = 4(-\frac{2}{17}) = -\frac{8}{17}$. The solution points are then $(1,0)$ and $(\frac{15}{17}, -\frac{8}{17})$.

9.
$$\frac{1}{1+x^3} = \frac{1}{(1+x)(1-x+x^2)} \equiv \frac{A}{1+x} + \frac{Bx+C}{x^2-x+1}.$$
Equating coefficients gives $A+B=0$, $B+C-A=0$, $A+C=1$. Hence $A=-B$,$C=-2B, C=1+B$ gives $B=-\frac{1}{3}$, $A=\frac{1}{3}$ and $C=\frac{2}{3}$. Therefore
$$\frac{1}{1+x^3} = \frac{1}{3(1+x)} + \frac{2-x}{3(x^2-x+1)}.$$

10.
$$\sqrt{3-\sqrt{5}} = \frac{\sqrt{a}-\sqrt{b}}{c} \Rightarrow 3-\sqrt{5} = \frac{a+b-2\sqrt{ab}}{c^2}.$$
Equating real and irrational parts gives the equations:
$$\frac{a+b}{c^2} = 3,\ \frac{4ab}{c^4} = 5 \Rightarrow \frac{a^2+b^2+2ab}{c^4} = 9$$
$$\Rightarrow \frac{a^2+b^2-2ab}{c^4} = 9-5 = 4 \Rightarrow \frac{a-b}{c^2} = \pm 2.$$
Now since $a > b$ we have $a-b = 2c^2$ and $a+b = 3c^2$ so that $a = \frac{5}{2}c^2$. Put $c=2$, then $a=10$ and $b = 12-10 = 2$. Hence
$$\sqrt{3-\sqrt{5}} = \frac{\sqrt{10}-\sqrt{2}}{2}.$$

Problem Set 6

1. Multiplying top and bottom by the conjugate of the denominator gives:
$$\frac{15+16i}{2+3i} = \frac{(15+16i)(2-3i)}{(2+3i)(2-3i)} = \frac{(30-48i^2)+i(32-45)}{2^2+3^2} = \frac{78-13i}{13} = 6-i.$$
.

2.
$$|z^{10}| = |z|^{10} = (\sqrt{1^2+7^2})^{10} = \sqrt{50}^{10} = 50^5 = 312,500,000.$$

3.
$$|\frac{-8\cdot 5+1\cdot 72i}{-1\cdot 72-8\cdot 5i}| = 1,$$
as in general $|\frac{z}{w}| = \frac{|z|}{|w|}$ and in this case these have the form
$$\frac{|a+bi|}{|-b+ai|} = \frac{\sqrt{a^2+b^2}}{\sqrt{b^2+a^2}} = 1.$$

4. Making x the subject of the equation we obtain

$$x(7+i-1)=22+3-2i \Rightarrow x=\frac{25-2i}{6+i}\cdot\frac{6-i}{6-i}=$$

$$\frac{(150-2)-i(25+12)}{36+1}=\frac{148-37i}{37}=4-i.$$

5. Put $z=x+iy$ and equate $z^2=(x^2-y^2)+2xyi=3-4i$. We get $xy=-2$. Substituting $y=-\frac{2}{x}$ gives

$$x^2-\frac{4}{x^2}=3 \Rightarrow x^4-3x^2-4=(x^2-4)(x^2+1)=0.$$

Since x is real, we get $x=\pm 2$ and $y=\mp 1$. Hence the roots in cartesian form are $\pm(2-i)$.

Comment Equating the second factor $x^2+1=0$ to give $x=\pm i$ will also yield the correct roots although the symbols x and y will no longer be real numbers, and so no longer represent the real and imaginary parts of those roots.

In polar form, write $z=re^{i\theta}$. The polar form of $z^2=r^2e^{2i\theta}$ is given from $r^4=3^2+(-4)^2=9+16=25$ (so that $r^2=5$ and $r=\sqrt{5}$); $\cos 2\theta=\frac{3}{5}$ and $\sin 2\theta=-\frac{4}{5}$. Now

$$\cos^2\theta=\frac{1}{2}(1+\cos 2\theta)=\frac{1}{2}(1+\frac{3}{5})=\frac{4}{5}\Rightarrow \cos\theta=\frac{\pm 2}{\sqrt{5}},$$

$$\sin^2\theta=\frac{1}{2}(1-\cos 2\theta)=\frac{1}{2}(1-\frac{3}{5})=\frac{1}{5}\Rightarrow \sin\theta=\pm\frac{1}{\sqrt{5}}.$$

We note that z^2 is in the 4th quadrant, so is one of its roots, showing the real and imaginary parts of z have opposite signs. The roots are therefore given by:

$$z=\pm\sqrt{5}(\frac{2}{\sqrt{5}}-\frac{i}{\sqrt{5}})=\pm(2-i).$$

6. Write $z=x+iy$. We then have

$$|z|-z=1-2i \Rightarrow |z|-1=z-2i=x+(y-2)i;$$

since the LHS is purely real we get $y-2=0$ so that $y=2$ and so $z=x+2i$. . Equating real parts in the previous equation gives

$$\sqrt{x^2+2^2}-1=x \Rightarrow x^2+4=(x+1)^2=x^2+2x+1$$

$$\Rightarrow 2x=3 \Rightarrow x=\frac{3}{2}. \text{ Hence}$$

$$z=\frac{3}{2}+2i.$$

7. Again writing $z=x+iy$ and working with the squares of the moduli gives the equation:

$$(x-1)^2+y^2=x^2+(y-3)^2 \Rightarrow x^2-2x+1+y^2=x^2+y^2-6y+9$$

$$\Rightarrow 6y - 2x - 8 = 0 \Rightarrow 3y - x = 4 \text{ or } y = \tfrac{1}{3}x + \tfrac{4}{3}$$

Comment This line is the perpendicular bisector of the line joining 1 to $3i$ in the complex plane.

8. Determine the region in the complex plane defined by

$$\text{Arg}(2 + iz) \leq \tfrac{\pi}{4}.$$

Write $z = x + iy$ then $w = 2 + iz = (2 - y) + ix$. Note that $z = \frac{w-2}{i}$. The region R representing points w that satisfy $\text{Arg}w \leq \frac{\pi}{4}$ consists of all those points on or below the ray $\theta = \frac{\pi}{4}$ and all points with negative polar angles (which does not include the negative real axis). To get the corresponding region S for z results from shifting R left by 2 followed by a rotation of $\frac{\pi}{2}$ clockwise. The ray $y = x$ $(x \geq 0)$ is transformed to $y = x + 2$ and then to $x + y = 2$. The negative real axis is transformed to the y-axis with $y > 2$. The region S is the region below the line $x + y = 2$ together with all z with non-positve real part, excluding the ray $\{z = iy : y > 2\}$.

9. Write $z = re^{i\theta}$ our equation becomes $r^3e^{3i\theta} = r^2$ so that either $r = 0$ or $re^{3i\theta} = 1$. Hence either $z = 0$ or $r = 1$ and $\theta \in \{0, \frac{2\pi}{3}, -\frac{2\pi}{3}\}$. Hence the solution set is

$$\{0, 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i\}.$$

10. We substitute $x = \frac{1}{2}(z + \overline{z})$ and $y = \frac{1}{2i}(z - \overline{z})$ into $(x + iy)^2 - (x + iy) + i$ to obtain

$$(x^2 - y^2 - x) + (2xy - y + 1)i = \frac{1}{4}\big(z^2 + \overline{z}^2 + 2z\overline{z} + z^2 + \overline{z}^2 - 2z\overline{z}\big) - \frac{1}{2}(z + \overline{z}) + \frac{1}{2}(z^2 - \overline{z}^2) - \frac{i}{2i}(z - \overline{z}) + i)$$

$$= z^2 - z + i.$$

Problem Set 7

1. $r^2 = (-2)^2 + (2\sqrt{3})^2 = 4 + 12 = 16 \Rightarrow r = 4$. Hence $\tan\theta = \frac{y}{x} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$. Since P lies in the second quadrant and $\tan\frac{\pi}{3} = \sqrt{3}$, we get $\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ and the polar cor-ordinates of P are $(4, \frac{2\pi}{3})$.

2. Putting $\theta = 0$ gives the maximum value of $r = 1 + 2 = 3$ and the point $(3, 0)$. At $\theta = \frac{\pi}{2}$ we get $r = 1$ and the point $(1, 0)$. Moreover $r = 0$ if and only if $\cos\theta = -\frac{1}{2} \Rightarrow \theta = \pm\frac{2\pi}{3}$; in particular $r \to 0$ as $\theta \to \frac{2\pi}{3}$. At $\theta = \pi$ we have $r = -1$, giving the point $(1, 0) = (-1, \pi)$ is also on the curve. Finally since the

cosine function is even, we infer that the graph is symmetric with respect to reflection in the polar axis.

3. The first maximum of $r = 5$ occurs when $\theta = \frac{\pi}{4}$ giving a leaf symmetric with respect to that ray. There are three copies of this leaf found by rotation through $\frac{\pi}{4}$ about the origin, the second of which appears in the *fourth* quadrant. Figure is traced out as θ ranges over $0 \leq \theta \leq 2\pi$.

4. Maximum of $\sqrt{2}a$ occurs at $\theta = 0$, with r going to 0 at $\theta = \frac{\pi}{4}$. Again picture symmetric with respect to polar axis. The propellor shape is traced out as θ passes through the range $0 \leq \theta \leq \pi$.

5.
$$r^4 = 2a^2r^2(\cos^2\theta - \sin^2\theta) \Rightarrow (x^2+y^2)^2 = 2a^2(x^2-y^2).$$

6. In cartesian form we have $(x-1)^2 + y^2 = 1^2$, which gives

$$(x^2+y^2) - 2x = 0 \Rightarrow r^2 - 2r\cos\theta = r(r - 2\cos\theta) = 0,$$
$$\therefore r = 2\cos\theta.$$

7. We have

$$\tan\theta = \frac{1+2x}{x} = 2 + \frac{1}{x} \Rightarrow x = \frac{1}{\tan\theta - 2}$$
$$\Rightarrow r\cos\theta = \frac{\cos\theta}{\sin\theta - 2\cos\theta} \Rightarrow r(\sin\theta - 2\cos\theta) = 1.$$

8.
$$[2(\cos 15^\circ + i\sin 15^\circ)]^4 = 2^4(\cos 60^\circ + i\sin 60^\circ) = 16(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 8 + 8\sqrt{3}i.$$

.

9. Write $z = x + iy$ with $z^2 = (x^2 - y^2 + 2xyi) = -i$ so that $x^2 - y^2 = 0$ and $2xy = -1$. This gives $y = \pm x$. If $y = x$ we get $2x^2 = -1$ has no solution in reals. If $y = -x$ we have $-2x^2 = -1$ so that $x = \pm\frac{1}{\sqrt{2}}$. Hence the two roots are $\pm\frac{1}{\sqrt{2}}(1-i)$. The fourth quadrant root is then $\frac{1}{\sqrt{2}}(1-i)$.

If we use polar form we write $-i = e^{-\frac{\pi}{2}i}$ giving the square roots, $e^{-\frac{\pi}{4}i}$ and $e^{\frac{3\pi}{4}i}$. The first root is that of the fourth quadrant and

$$e^{-\frac{\pi}{4}i} = (\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(1-i).$$

.

10. $-8 = 8e^{i\pi}$ so by De Moivre's Theorem, in polar form the cube roots are:

$$\{2e^{\frac{\pi}{3}i},\, 2e^{\pi i},\, 2e^{\frac{5\pi}{3}}\} = \{2(\frac{1}{2} + i\frac{\sqrt{3}}{2}),\, 2(-1+0i),\, 2(-\frac{1}{2} + i\frac{\sqrt{3}}{2})\}$$
$$= (1 + i\sqrt{3},\, -2,\, -1 + i\sqrt{3}).$$

Problem Set 8

1. With an obvious notation, we have the following equations:

$$M + W + C = 20$$

$$3M + 2W + \frac{1}{2}C = 20.$$

I here solve these equations by matrix elimination but this corresponds to dividing the second equation by 3 and subtracting to eliminate the Men:

$$\begin{bmatrix} 1 & 1 & 1 & 20 \\ 3 & 2 & \frac{1}{2} & 20 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 20 \\ 0 & -1 & -\frac{5}{2} & -40 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{3}{2} & -20 \\ 0 & -1 & -\frac{5}{2} & -40 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -\frac{3}{2} & -20 \\ 0 & 1 & \frac{5}{2} & 40 \end{bmatrix}, \text{ yielding the system}$$

$$M = \frac{3}{2}C - 20;\ W = 40 - \frac{5}{2}C.$$

No M, C, W are positive integers, so we must have $C = 2A$, for some positive integer A, whence:

$$M = 3A - 20 > 0 \Leftrightarrow A > \frac{20}{3} \Leftrightarrow A \geq 7;$$

$$W = 40 - 5A > 0 \Leftrightarrow A < 8 \Leftrightarrow A \leq 7.$$

Hence $A = 7$, and so $C = 14, W = 5, M = 1$.

Comment If we allow some of C, W and M to be 0, then there is an alternative solution in $C = 16, M = 4$ and $W = 0$.

2. Let A denote the number of streets/hour that A can deliver. The respective joint rates in units of streets/hour of the three types of teams are given as:

$$A + B = 4$$

$$B + C = 6$$

$$A + C = 5.$$

Summing the three equations gives $2A+2B+2C = 15 \Rightarrow A+B+C = \frac{15}{2} = 7{\cdot}5$ streets/hour. Hence the time they would take all working together for a single street in minutes is $\frac{2}{15}$ hours, which is

$$60 \times \frac{2}{15} = 8 \text{ minutes}.$$

3.

$$3x^2 - 6x + 3 = 2yy\prime \Rightarrow y\prime = \frac{3x^2 - 6x + 3}{2y};$$

so that $y\prime(0) = \frac{3}{2}$. The equation of the tangent is thus $y = \frac{3}{2}x + 1$. Substituting accordingly gives:

$$x^3 - 3x^2 + 3x + 1 = \frac{9}{4}x^2 + 3x + 1 \Leftrightarrow 4x^3 - 12x^2 = 9x^2$$

$$\Leftrightarrow x^2(4x - 21) = 0;$$

the non-zero root of which is $x = \frac{21}{4}$, whence $y = \frac{3}{2} \cdot \frac{21}{4} + 1 = \frac{71}{8}$. This gives a new rational point on the curve:

$$(x, y) = \left(\frac{21}{4}, \frac{71}{8}\right).$$

Comment This problem used differentiation and indeed implicit diffierentiation to find the tangent. The reason why this works is that first, since the point is rational, the coefficients in the tangent are also rational. Since the line is a tangent (and not just an intersecting line) the root of the equation that arises when we equate tangent and curve at this point is of order 2, so that the root $x = 0$ is a double root. Since the repeated root 0 represents two roots of the corresponding cubic equation, the third root is also real and is indeed rational as an irrational root would lead to a linear factor that would give irrational coefficients in the polynomial when the linear factorization was expanded, which is not the case as the coefficients of the given cubic equation are rational.

4. Let x denote Diophantus's life span. Then the given information is captured by equating his son's lifespan to half of his father's:

$$x - \left(\frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + 4\right) = \frac{x}{2}$$

$$\Leftrightarrow x\left(\frac{1}{2} - \frac{1}{6} - \frac{1}{12} - \frac{1}{7}\right) = 9 \Rightarrow x\left(\frac{42 - 14 - 7 - 12}{84}\right) = 9$$

$$\Rightarrow x = \frac{84}{9} \cdot 9 = 84\,\text{years}.$$

5. Putting $u = \frac{1}{x}$ and $v = \frac{1}{y}$ we obtain the equations $u - 3v = 2$ and $2u + v = 3$, which gives:

$$2u - 6v = 4,\ 2u + v = 3.$$

Subtracting the first equation from the second gives $7v = -1$ so that $v = -\frac{1}{7}$ and so $u = 2 + 3v = 2 - \frac{3}{7} = \frac{11}{7}$.

$$\therefore x = \frac{1}{u} = \frac{7}{11},\ y = \frac{1}{v} = -7, \text{ so that } (x, y) = (\tfrac{7}{11}, -7).$$

6.

$$\begin{aligned} x + 3y + 3z &= 1 \\ 6x - 2y - 2z &= 4 \\ \Leftrightarrow x + 3y + 3z &= 1 \\ 9x - 3y - 3z &= 6; \end{aligned}$$

and adding the equations eliminates both y and z, giving $10x = 7$ so that $x = \frac{7}{10}$.

Comment It sometimes surprises that the exact value of some of the unknowns can be found in an under determined system such as this one. It is possible however and the phenomenon can be made more visible with examples such as the 'system' in three unknowns $x = y = 0$—clearly the values of x and y are known while z is left completely 'free', the solution in this case consisting of all points on the z-axis.

7. Let w denote the width of the river. When the boats first pass, one of them has travelled 720 yards. When they next meet, that same boat has travelled $w+400$ yards. When the ferries first meet the *total* distance they have travelled is w. When they next meet the total distance travelled is $3w$ (each has made one complete crossing and the the two partial crossings again sum to w). Since the ferries move at constant (albeit different) speeds, each has travelled three times as far when they meet for the second time as when they first crossed. We infer the equation:

$$w + 400 = 3 \times 720 \Rightarrow w = 2160 - 400 = 1760 \text{ yards.}$$

Therefore Sam Loyd's river is exactly 1 mile wide.

Comment Note that the length of the changeover period did not enter into the reckoning. It would only matter if it were different for the two ferries, in which case we would need more information in order to solve the problem.

8. Let w denote the width of the river and let us work generally, letting a and b stand for the respective distances from the left and right banks at the times of the first and second crossovers as in our diagram of the previous page. Let u be the speed of the ferry that initially departs from the left bank and let v be the speed of the second boat. If we let t be the time of the first meeting we have: $a = ut,\ w - a = vt$ and making t the subject of each equation gives:

$$\frac{a}{u} = \frac{w-a}{v} \Rightarrow \frac{a}{w-a} = \frac{u}{v}.$$

Letting t now denote the travelling time until the second rendevous (ignoring the inconsequential stopovers) we get $w + b = ut,\ 2w - b = vt$ and so

$$\frac{w+b}{u} = \frac{2w-b}{v} \Rightarrow \frac{w+b}{2w-b} = \frac{u}{v}.$$

Equating the two expressions for $\frac{u}{v}$ now gives:

$$\frac{a}{w-a} = \frac{w+b}{2w-b} \Rightarrow 2aw - ab = w^2 + bw - aw - ab$$

$$\Rightarrow w^2 + (b-3a)w = 0;$$

since $w \neq 0$ we may divide this final equation by w and thus obtain $w = 3a - b$.

Returning to the original problem, where a and b are given as $a = 720$ and $b = 400$ respectively, we recover $w = 3 \times 720 - 400 = 2160 - 400 = 1760$. At the same time we have that the ratio of the boat speeds, $\frac{u}{v}$, is given by

$$\frac{u}{v} = \frac{a}{w-a} = \frac{720}{1760-720} = \frac{720}{1040} = \frac{9}{13}.$$

In words, the ratio of the speed of the slower to the faster ferry is 9 : 13.

9. Let us take the origin, O, from which we measure the position of both man and bus to be the man's initial position as the bus begins to move. At time t he will have travelled a distance vt in the positive x-direction. The *speed* of the bus at time t however is at. Since it started from rest and the acceleration is constant, the average speed of the bus in the time interval from time 0 up to time t is $\frac{at-0}{2} = \frac{1}{2}at$. Hence the distance of the door of the bus from the origin at time t is $d + \left(\frac{1}{2}at\right)t = d + \frac{1}{2}at^2$.

Now the runner will be at the same position as the door at the times when these two expressions for the position of man and bus agree, which is to say when time t satisfies:

$$vt = d + \frac{1}{2}at^2 \Rightarrow 2vt = 2d + at^2$$

$$\Rightarrow at^2 - 2vt + 2d = 0. \quad (4)$$

The question asks us then to find, for a given value of a, what is the value of d where there is just one solution to (4). To determine this value, we just need to set the discriminant $\Delta = 0$. First let's find Δ for our equation (4):

$$\Delta = (-2v)^2 - 4a(2d) = 4v^2 - 8ad;$$

putting $\Delta = 0$ now gives the critical value of d:

$$8ad = 4v^2 \text{ and so } d = \frac{4v^2}{8a} = \frac{v^2}{2a}.$$

The man will catch his bus as along as $d \leq \frac{v^2}{2a}$. For example, if the man is running at 6m/sec and the bus is accelerating at 1m/sec^2 then the critical value is $d = \frac{6^2}{2(1)} = 18$, so if the initial gap between him and the door is more than 18m, then the bus gets away.

10. Let $x =$ current age of the ship, $y =$current age of the boiler. Let z be the number of years ago when the ship was as old as the boiler is now. Then

$$x - z = y \Rightarrow z = x - y.$$

Hence

$$x = 2(y - z) = 2(y - (x - y)) = 2(2y - x) = 4x - 2x$$
$$\Rightarrow 3x = 4y, \text{ so that the ratio } x : y \text{ is } 4 : 3.$$

Problem Set 9

1. $a = t_3 - 2d = 4 - 2(3) = -2$. Hence $t_n = a + (n-1)d = -2 + 3(n-1) = 3n - 5$.

2. The sum of the first n terms is $na + \frac{d}{2}n(n-1)$ so the sum of the first $2n$ terms is:

$$2na + \frac{d}{2}(2n)(2n-1) = -4n + 3n(2n-1) =$$
$$= -4n + 6n^2 - 3n = 6n^2 - 7n = n(6n-7).$$

3. This is a geometric progression with $a = 2$ and $r = \frac{1}{2}$; hence the sum of the first 12 terms is:

$$\frac{2\big(1 - (\frac{1}{2})^{12}\big)}{1 - \frac{1}{2}} = 4 - \big(\frac{1}{2}\big)^{10} = 3\,\frac{1023}{1024}.$$

4.

$$\sum_{n=1}^{\infty}(n-2)\big(\frac{1}{3}\big)^n = -\frac{1}{3} + 0 + \frac{1}{9}\sum_{n=3}^{\infty}(n-2)\big(\frac{1}{3}\big)^{n-2}$$
$$= -\frac{1}{3} + \frac{1}{9}\sum_{m=1}^{\infty} m \cdot \big(\frac{1}{3}\big)^m, \text{ where } m = n-2$$

Now put $x = \frac{1}{3}$ into the given formula:

$$= -\frac{1}{3} + \frac{1}{9} \cdot \frac{1/3}{(2/3)^2} = -\frac{1}{3} + \frac{1}{12} = \frac{1-4}{12} = -\frac{3}{12} = -\frac{1}{4}.$$

Comment: the formula can be got by term-by-term differentiating of the standard geometric summation:

$$\sum_{n=0}^{\infty} x^n = (1-x)^{-1} \Rightarrow \sum_{n=0}^{\infty} nx^{n-1} = (1-x)^{-2}$$

$$\Rightarrow \sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}, \ \ |x| < 1.$$

5. The common difference d of the arithmetic progression is:

$$d = \frac{t_n - t_m}{n - m} = \frac{m - n}{n - m} = -1.$$

Hence $t_{n+m} = t_n + md = m + m(-1) = m - m = 0$.

Comment This was another problem set for me by my old schoolteacher, Mr Wiseman, in 1972.

6.

$$S_n = r(1 + r^2 + r^4 + \cdots + (r^2)^{n-1}) = \frac{r(1 - r^{2n})}{1 - r^2} \text{ or } \tfrac{r(1-r^n)(1+r^n)}{(1-r)(1+r)}.$$

Now $\lim_{n\to\infty} \frac{r(1-r^{2n})}{1-r^2} = \frac{r}{1-r^2}$ if $|r| < 1$. Put $r = \frac{1}{2}$ to get

$$S_n \to \frac{(\frac{1}{2})}{1 - \frac{1}{4}} = \frac{2}{3}.$$

7.

$$\sum_{m=1}^{n} [(m+1)^2 - m^2] = \sum_{m=1}^{n} (2m+1) \Rightarrow (n+1)^2 - 1^2 = 2\sum_{m=1}^{n} m + \sum_{m=1}^{n} 1$$

$$\Rightarrow n^2 + 2n + 1 - 1 = \big(2\sum_{m=1}^{n} m\big) + n$$

$$\Rightarrow \sum_{m=1}^{n} m = \frac{n^2 + n}{2} = \frac{1}{2}n(n+1).$$

8.

$$\sum_{m=1}^{n} [(m+1)^3 - m^3] = 3\sum_{m=1}^{n} m^2 + 3\sum_{m=1}^{n} m + \sum_{m=1}^{n} 1$$

$$\Rightarrow (n+1)^3 - 1^3 = 3\sum_{m=1}^{n} m^2 + 3\sum_{m=1}^{n} m + \sum_{m=1}^{n} 1$$

$$\Rightarrow (n+1)^3 - 1^3 = 3\sum_{m=1}^{n} m^2 + \frac{3}{2}n(n+1) + n$$

$$\Rightarrow 3\sum_{m=1}^{n} m^2 = n^3 + 3n^2 + 3n - \frac{3}{2}n^2 - \frac{3}{2}n - n$$

$$\Rightarrow \sum_{m=1}^{n} m^2 = \frac{1}{3}\big(n^3 + \frac{3}{2}n^2 + \frac{n}{2}\big)$$

$$\therefore \sum_{m=1}^{n} m^2 = \frac{n}{6}\big(2n^2 + 3n + 1\big) = \frac{n}{6}(n+1)(2n+1).$$

Comment It can now be seen that this technique can be used recursively to find a closed expression for the sum of cubes, the sum of fourth powers, and so on.

9. Put $a = -1$ and $b = 1$ in the Binomial theorem:

$$0 = (-1+1)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k \cdot (1)^{n-k}$$

$$\Rightarrow \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Comment Putting $a = b = 1$ on the other hand yields

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n,$$

which can be interpreted as saying that the total number of sets of a set of order n is 2^n.

10. Put

$$\frac{1}{n(n+1)} \equiv \frac{A}{n} + \frac{B}{n+1} = \frac{A(n+1) + Bn}{n(n+1)};$$

putting $n = 0$ gives $A = 1$, putting $n = -1$ gives $B = -1$ and so $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ (or use the so-called *Cover Up Method*). Hence for any partial sum up to N we have:

$$\sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \Big(\frac{1}{n} - \frac{1}{n+1}\Big) = 1 - \frac{1}{N+1};$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n(n+1)} = \lim_{N\to\infty} \Big(1 - \frac{1}{N+1}\Big)$$

$$1 - \lim_{N\to\infty} \Big(\frac{1}{N+1}\Big) = 1 - 0 = 1.$$

Problem Set 10

1. We need the conversion formula from degrees Fahrenheit F to degrees Celsius C, which is $C = \frac{5}{9}(F - 32)$. We put $C = F$ and get:

$$C = \frac{5}{9}C - \frac{5}{9} \cdot 32 \Rightarrow C\Big(1 - \frac{5}{9}\Big) = -\frac{5}{9} \cdot 32$$

$$\Rightarrow \frac{4}{9}C = -\frac{5}{9} \cdot 32 \Rightarrow C = -5 \cdot \frac{32}{4} = -5 \cdot 8 = -40.$$

Therefore the two scales agree at just one value: $-40°$.

2. Writing the daily milk output of the black and brown cows as Bl and Br respectively, we capture all the given information in the single equation:

$$5(4Bl + 3Br) = 4(3Bl + 5Br)$$

$$\Leftrightarrow 20Bl + 15Br = 12Bl + 20Br$$

$$\Leftrightarrow 8Bl = 5Br \Leftrightarrow Br = \frac{8}{5}Bl,$$

so that it is the Browns that are superior.

3. The ratio of the slow to the fast runner's speed is $60 : 100 = 3 : 5$. Hence when the faster athlete has run a distance d the slower one has run only $\frac{3}{5}d$. Given the fact the slow man had 100 paces start, the distance when the fast man catches up satisfies the equation:

$$d = 100 + \frac{3d}{5} \text{ and so } \frac{2d}{5} = 100$$

$$\Rightarrow d = \frac{5}{2} \times 100 = 250.$$

Therefore the fast runner will have travelled 250 paces when he catches the slower runner.

4. Working in units of millions and letting r and s denote the number of rentals and of sales respectively we have the equations:

$$r + s = 2$$

$$6r + 15s = 15.$$

Multiplying the first equation by 6 and subtracting the result from the second equation then gives:

$$6r + 6s = 12 \Rightarrow 9s = 3 \text{ and so } s = \frac{1}{3}, \ r = 2 - s = 2 - \frac{1}{3} = \frac{5}{3}.$$

Therefore there were $\frac{1}{3}$ million sales and $1\frac{2}{3}$ million rentals.

Comment The information in the question appeared in a media report, along with the comment that the company had not said how many rentals and how many were sales. As we can see, they should have been able to figure that out!

5.
$$0 = 1 + x + x^2 + x^3 = \frac{1 - x^4}{1 - x} \ (x \neq 1). \text{ Hence}$$
$$x^4 = 1 \Rightarrow x \in \{-1, i, -i\}.$$

6. For the series to converge we must have $|x| < 1$ in which case
$$x^2 + x^3 + x^4 + \cdots = \frac{x^2}{1-x} = 1 \Rightarrow \frac{x^2 - (1-x)}{1-x} = 0$$
$$\Rightarrow x^2 + x - 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1+4}}{2} = z_1 = \frac{-1+\sqrt{5}}{2} \text{ or } z_2 = \frac{-1-\sqrt{5}}{2}.$$
Now $|z_1| < 1$ but $|z_2| > 1$ so the series fails to converge for z_2. Therefore the unique solution is $\frac{-1+\sqrt{5}}{2}$.

7. Let a be the unknown initial number of potatoes. The sequence follows a geometric progression with $r = 1 - \frac{1}{3} = \frac{2}{3}$. We want the 4th term, which is
$$ar^3 = 8 \Rightarrow a = \frac{8}{r^3} = 8 \cdot \frac{3^3}{2^3} = 27.$$

8. Applying the Binomial expansion for the power $n = 3$ we obtain:

$(y+t)^3 + a(y+t)^2 + \cdots = 0$ and so $y^3 + (3t+a)y^2 +$ (terms in y and constants) $= 0$.

Therefore if we put $t = -\frac{a}{3}$ the outcome will be a monic cubic in y with no term in y^2.

Comments It follows that we will be able to solve any cubic provided that we learn how to solve a *depressed cubic*, which is to say one of the form $x^3 + dx + e = 0$, for the general cubic can be reduced to one of this kind.

9. Applying Vieta's substitution, $x = v + \frac{s}{v}$ gives:
$$(v + \frac{s}{v})^3 + d(v + \frac{s}{v}) + e = v^3 + 3sv + \frac{3s^2}{v} + \frac{s^3}{v^3} + dv + \frac{ds}{v} + e.$$
$$= v^3 + (3s + d)v + (3s + d)\frac{s}{v} + \frac{s^3}{v^3} + e.$$
By putting $s = -\frac{d}{3}$, for then *both* the terms in v and in $1/v$ vanish, leaving the equation:
$$v^3 - \frac{d^3}{27v^3} + e = 0 \text{ whence } v^6 + ev^3 - \frac{d^3}{27} = 0.$$

Finally, by substituting $z = v^3$ we reduce the problem to the quadratic equation $z^2 + ez - \left(\frac{d}{3}\right)^3 = 0$.

10. To get a depressed cubic we put $x = y + t$ where $t = -\frac{a}{3} = -\frac{(-3)}{3} = 1$, so we put $x = y + 1$, which gives

$$(y+1)^3 - 3(y+1)^2 + 6(y+1) + 8 = (y^3 + 3y^2 + 3y + 1) - (3y^2 + 6y + 3) + (6y + 6) + 8 = 0$$

$$\Rightarrow y^3 + 3y + 12 = 0.$$

Next we use the substitution: $y = v + \frac{s}{v} = v - \frac{d}{3}v = v - \frac{3}{3v} = v - \frac{1}{v}$, which gives

$$\left(v - \frac{1}{v}\right)^3 + 3\left(v - \frac{1}{v}\right) + 12 = v^3 - 3v + \frac{3}{v} - \frac{1}{v^3} + 3v - \frac{3}{v} + 12 = 0$$

$$\Rightarrow v^3 - \frac{1}{v^3} + 12 = 0 \Rightarrow v^6 + 12v^3 - 1 = 0.$$

Putting $z = v^3$ gives the quadratic

$$z^2 + 12z - 1 = 0$$

$$\Rightarrow z = \frac{-12 \pm \sqrt{144 + 4}}{2} = \frac{-12 \pm \sqrt{148}}{2} = \frac{-12 \pm \sqrt{4 \times 37}}{2} = -6 \pm \sqrt{37}.$$

Taking the root $\sqrt{37} - 6$, we get that $v = \sqrt[3]{\sqrt{37} - 6}$ (the alternative choice leads to the same set of roots). Continuing, we now find a real root:

$$x = y + 1 = v - \frac{1}{v} + 1 = \frac{v^2 + v - 1}{v}.$$

So, finally, we have our real root:

$$x = \frac{(\sqrt{37} - 6)^{\frac{2}{3}} + (\sqrt{37} - 6)^{\frac{1}{3}} - 1}{(\sqrt{37} - 6)^{\frac{1}{3}}} \approx -0 \cdot 8589.$$

Comment: Returning to the original polynomial $p(x) = x^3 - 3x^2 + 6x + 8$, we may check that $p(-1) = -2 < 0 < 8 = p(0)$ and so there must be a value of x in the interval $-1 < x < 0$ such that $p(x) = 0$. What is more, the values of $p(-1)$ and $p(0)$ suggest that the root lies closer to -1 than to 0, which is what has transpired.