Mathematics 104 Numbers & Discrete Mathematics Solutions

Professor Peter M. Higgins

November 9, 2019

Solutions and Comments for the Problems

Problem Set 1

1. We take the hint and write $a = 0\cdot\dot{6}\dot{3}$. Then $100a - a = 63\cdot\dot{6}\dot{3} - 0\cdot\dot{6}\dot{3} = 63$. Hence

$$a = \frac{63}{99} = \frac{7}{11}.$$

Comment We shall see this kind of trick later on as well to simplify other infinite repeating processes. A more prosaic point is that students should not forget to cancel down a fraction if possible.

2. $11010110_2 = 2^1 + 2^2 + 2^4 + 2^6 + 2^7 = 2 + 4 + 16 + 64 + 128 = 214.$

3. $100 = 64 + 32 + 4 = 2^6 + 2^5 + 2^2$ so that $100 = 1100100_2$.

4. Following the method of Question 1 for base 3, we write $a = (0\cdot\overline{20})_3$, so that $100_3 a - a = 20\cdot\overline{20}_3 - 0\cdot\overline{20}_3 = 20_3$. Hence

$$a = (\frac{20}{22})_3 = (\frac{10}{11})_3 = \frac{3}{4}.$$

5. Following the hint we compute:

$$\frac{9}{20} - \frac{1}{3} = \frac{27-20}{60} = \frac{7}{60}, \ \frac{7}{60} - \frac{1}{10} = \frac{7-6}{60} = \frac{1}{60},$$

so that

$$\frac{9}{20} = \frac{1}{3} + \frac{1}{10} + \frac{1}{60}.$$

Comment This technique of subtracting the largest reiprocal possible will always yield an Egyptian decomposition as it can be shown that the numerator of the fraction remaining after each subtraction is less than before. The number k that is the denominator of the largest reciprocal less than the given fraction $\frac{m}{n}$ is given by $k = \lceil \frac{n}{m} \rceil$. Another technique however for finding these decompositions is that of the *Akhmim papyrus* (6th century AD), which is based on applying the identity:

$$\frac{m}{pq} = \frac{m}{p(p+q)} + \frac{m}{q(p+q)}.$$

Applying this approach to the fraction $\frac{9}{20}$ yields the two-fraction decomposition $\frac{1}{4} + \frac{1}{5}$.

6.

$$1729 = 1^3 + 12^3 = 9^3 + 10^3.$$

Comment This example is always talked about because, as Ramanujan pointed out to Hardy in a conversation about a taxi cab number, 1729 is the smallest number that is the sum of two cubes in two different ways.

7. Let a be the number that agree with the proposal and let b be the sample size. Then we are told that

$$\frac{100a}{b} = 76.8 \Rightarrow \frac{a}{b} = \frac{76.8}{100} = \frac{768}{1,000}.$$

The smallest the numbers in this ratio can be, given that a and b are positive integers, is found by cancelling this fraction to its reduced form, which gives $\frac{96}{125}$. Hence the smallest the sample size could be is 125 (of which, 96 agreed with the proposal and $125 - 96 = 29$ did not).

8. The first three primes are $2, 3$ and 5. We find in each case that $2^p - 1$ is a prime as we get $3, 7$ and 31 respectively. Hence applying the formula of Euclid we get the three perfect numbers $2^{p-1}(2^p - 1)$ as being for

$$p = 2: \ 2^1(2^2-1) = 2\times3 = 6, \ p = 3: \ 2^2(2^3-1) = 4\times7 = 28, \ p = 5: \ 2^4(2^5-1) = 16\times31 = 496.$$

9. The prime factorization of 220 is $2^2 \times 5 \times 11$ and so the sum of its factors is

$$1 + (2 + 5 + 11) + (4 + 10 + 22 + 55) + (20 + 44 + 110) = 1 + 18 + 91 + 64 = 284.$$

On the other hand the prime factorization of 284 is $2^2 \times 71$ and so the sum of its factors is

$$1 + (2 + 71) + (4 + 142) = 1 + 73 + 146 = 220.$$

Therefore $(220, 284)$ is indeed an amicable pair.

Comment Indeed this is the smallest amicable pair. Another small amicable pair is $(1184, 1210)$ found by 16-year-old Nicolo Paganini in 1866.

10. FRED, EATS $< 10,000$, ADDER $> 10,000 \Rightarrow A = 1$.

If $T = 0$ then there is no carry from column 1 and no carry from column 3. Then either $R + 1 = D$ or $R = 9, D = 0$. If $R + 1 = D$ then $R < D$, but $R = D + S$! (contradiction). If $R = 9, D = 0$ then $0 + S = R \Rightarrow S = R$! Hence $T \neq 0$, so $T = 9$, and there is a carry from column 1.

Thus $R < D$, there is carry to column 3, so $R + 2 = D$. $D + S = 10 + R \Rightarrow R + 2 + S = 10 + R \Rightarrow S = 8$. If $R = 0$ then $D = 2$. $E + F = 12, E \neq F$; $E \neq 0, 1, 2, 3$ (for otherwise $F = 9$!) or 4(otherwise $F = 8$!). If $E = 5$ then $F = 7$. If $E = 6 \Rightarrow E = F$! If $E = 7$ then $F = 5$; $E \neq 8, 9$.

Now $R \neq 1$ (as $A = 1$). If $R \geq 2 \Rightarrow D \geq 4 \Rightarrow E + F \geq 14$. Hence $E \neq 0, 1, 2, 3, 4$; $E \neq 5$ (as $F \neq 9$) $E \neq 6$ (as $F \neq 8, 9$), $E \neq 7$ (as $E \neq F$ and $F \neq 8, 9$); $E \neq 8, 9$. Therefore there are just the two solutions:

$$7,052 + 5,198 = 12,250; \ 5,072 + 7,198 = 12,270.$$

Problem Set 2

1. $\ln(10^6) = 6\ln 10 \approx 19.8$ and so 20 is the least positive integer k such that $2^k \geq 10^6$. Hence, in binary, 10^6 has 20 digits.

2. $\frac{1}{4} = 2^{-2}$ so that as a binary 'decimal', $\frac{1}{4} = 0 \cdot 01$.

3. Any integer can be written uniquely in binary, which is to say as a sum of powers of 2. Hence $B = \{1, 2, 4, 8, 16, 32, 64\}$ is a set of order 7 and any number up to the sum of its elements, which is $2^7 - 1 = 127$, can be written as a sum of some subset of B.

4. We require

$$a^2 + (p^2 - q^2)^2 = (p^2 + q^2)^2 \Rightarrow a^2 = 4p^2q^2 \Rightarrow a = 2pq.$$

5. We put $p^2 + q^2 = 17$, so we take $p = 4$ and $q = 1$. Then $p^2 - q^2 = 16 - 1 = 15$ and $2pq = 8$. The trio of sides for the required right-angled triangle is $(8, 15, 17)$.

6.

$$\lceil \frac{13}{6} \rceil = 3, \ \frac{6}{13} - \frac{1}{3} = \frac{18-13}{39} = \frac{5}{39};$$
$$\lceil \frac{39}{5} \rceil = 8, \ \frac{5}{39} - \frac{1}{8} = \frac{40-39}{312} = \frac{1}{312};$$
$$\therefore \frac{6}{13} = \frac{1}{3} + \frac{1}{8} + \frac{1}{312}.$$

7. $\{\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}\}$.

8 & 9. Since \$1 = 100 cents and all coins except the penny are multiples of 5, the number of pennies must be a multiple of 5. If that number were 35, the remaining coins would total at least $(50-35)\times 5 = 15\times 5 = 75$ and $35+75 = 110$, exceeding the target total. Clearly if the number of pennies were less than 35 the target would drift further out of reach. On the other hand 50 pennies won't work and therefore the number of pennies in any solution is 40 or 45.

If we try 40 pennies then the remaining 10 coins sum to $100 - 40 = 60$c. By the same argument as in the previous paragraph we see that the number of nickels (5c coins) must be at least 8, but 9 or 10 nickels do not provide a solution. With 8 nickels there is $60 - (8 \times 5) = 60 - 40 = 20$c remaining to be made up of two coins, which then must be 2 dimes (10c coins). This gives the first solution:

$$(40 \times 1) + (8 \times 5) + (2 \times 10) = 100.$$

Next, let us examine the alternative possibility that the number of pennies is 45, which requires us to make up $100 - 45 = 55$c from $50 - 45 = 5$ coins. If the number of nickels is even, we require 1 quarter to give 55 (as 55 is an odd multiple of 5). Hence we need to make up $(55-25) = 30$c from the 4 remaining coins. Clearly this is only possible with 2 nickels and 2 dimes, giving a second solution:

$$(45 \times 1) + (2 \times 5) + (2 \times 10) + (1 \times 25) = 100.$$

If on the other hand the number of nickels were odd, then we cannot use any quarters and we would have to compile 55c from 5 coins, each of which is a nickel or a dime. However, if there were just a single nickel, we would need 5 dimes to make 55c, while if we try 3 nickels we need 4 dimes, while 5 (or more) nickels are clearly impossible. Hence there is no other solution to this problem apart from the two that we have identified.

10. Each man has $\frac{8}{3} = 2\frac{2}{3}$ loaves, costing the hunter 8 piasters, that is $8 \div \frac{8}{3} = 3$ piasters/loaf. The shepherd with 3 loaves gave away $\frac{1}{3}$ of a loaf so is owed $\frac{1}{3} \cdot 3 = 1$ piaster. The other gets $8 - 1 = 7$ piasters (having given the hunter $2\frac{1}{3} = \frac{7}{3}$ loaves worth $\frac{7}{3} \cdot 3 = 7$ piasters).

Problem Set 3

1. $146 = 3^4 + 2 \cdot 3^3 + 3^2 + 0 \cdot 3^1 + 2 \cdot 3^0 = (12102)_3$.

2. The sum of the vertices is $1 + 2 + \cdots + 8 = \frac{1}{2} \cdot 8 \cdot 9 = 36$. Each vertex contributes to 3 faces of the cube so the total sum of the faces is 3×36. Since each face has the same sum, that common sum is $\frac{3 \times 36}{6} = 18$.

3. The sum of the edges is $1 + 2 + \cdots + 12 = \frac{1}{2} \cdot 12 \cdot 13 = 78$. Each edge contributes to 2 faces and so the common edge sum of the faces is $\frac{2 \times 78}{6} = \frac{78}{3} = 26$.

Comment And it is possible to find a number of solutions to the problems of Question 2 and 3.

4. We see that ϕ must satisfy

$$\phi = 1 + \frac{1}{\phi} \Rightarrow \phi^2 - \phi - 1 = 0 \Rightarrow \phi = \frac{1 \pm \sqrt{1+4}}{2},$$

and since it is also evident that $\phi > 0$ we conclude that $\phi = \frac{1+\sqrt{5}}{2}$, the golden ratio.

5. Up to and including 1000 there are $\frac{1000}{2} = 500$ even numbers, $\lfloor \frac{1000}{3} \rfloor = 333$

multiples of 3, and $\lfloor\frac{1000}{6}\rfloor = 166$ multiples of 6. Hence the total of all numbers not divisible by either 2 or 3 in this range is

$$1000 - (500 + 333 - 166) = 1000 - 667 = 333.$$

6. $7^4 = 2401 \equiv 1 \pmod{100}$. Now $355 = (88 \times 4) + 3$ so we get

$$7^{355} = (7^4)^{88} \cdot 7^3 \equiv 1^{88} \cdot 343 \equiv 43 \pmod{100}.$$

Hence the final two digits of 7^{355} are 43.

7. All factorials $n!$ where $n \geq 10$ contain the factors $2, 5$ and 10 and thus are divisible by $2 \times 5 \times 10 = 100$ and so their final two digits are 00. Hence

$$0! + 5! + 10! + \cdots + 100! \equiv 1 + 120 + 0 \pmod{100} \equiv 21 \pmod{100}.$$

Hence the final two digits of this sum are 21.

8. The highest prime powers up to 10 are $2^3 = 8$, $3^2 = 9$, 5, and 7. Hence the smallest number divisible by all integers from 1 to 10 is their product $8 \times 9 \times 5 \times 7 = 2520$.

9. Let $L = \{50, 4 \times 20, 1 \times 5, 4 \times 2\}$. The sum of L is £1.43. It is not possible to get £1 from L, for suppose this were possible. We must include the 50p (as the remainder only make 93p). Since $50 + 5 + 8 = 63$ we must have at least 2 x 20p, and not more than 2 x 20p as we must include 50p. We then have $50 +$ 2 x $20 = 90$p and the remaining 10p cannot be made up from the smaller coins. Hence L is a solution.

Now let C be a maximal collection that can't make £1. Suppose that C sums to at least £1.43. We show that in this case $C = L$, which completes the proof.

Clearly C has no more than 1 x 50p and no more than 4 x 20p.

If C has more than 1 x 10p, we can take one 10p and pair off the remaining 10p pieces, replacing each pair by 20p to give a new collection that has the same total value and also cannot make £1. (For if we could make £1 with the new collection, by replacing 20p coins by pairs of 10p's, we could make £1 from the original C, which we cannot.) The same argument applies to multiple 5p pieces, (replacing pairs by 10p pieces) so we may assume that C has at most 1 x 10p and at most 1 x 5p. By the same token, we may assume that there are no more than 4 x 2p pieces (as we could replace 5 x 2 by a 10) and at most 1 x 1p piece (as pairs of 1p's could be replaced by 2p's).

This means that C is contained in the set $\{50, 4 \times 20, 10, 5, 4 \times 2, 1]$. However, $50 +$ 4 x $20 + 10 = 100$, so at least one of these coins must be deleted from C. If we delete 50p then C sums to only £1.04, so 50 is included. If we have no more than 3 x 20, then C sums to only £1.34, so that C also contains 4 x 20, and so not 10p. Hence either $C = L$ or $C = L + 1p$, but the latter is not possible as it gives 50 + (2 x 20) + 5 + (2 x 2) + 1 = £1.00. Hence $C = L$, as required.

10. From column 5, $M = 1$ since it is the only carry-over possible from the sum of two single-digit numbers in column 4. Since there is a carry in column 5, O must be less than or equal to M (from column 4). But O cannot be equal to M, so O is less than M. Therefore $O = 0$. Since O is 1 less than M, S is either 8 or 9 depending on whether there is a carry in column 4. But if there were a carry in column 4, N would be less than or equal to O (from column 3). This is impossible since O = 0. Therefore there is no carry in column 4 and $S = 9$. If there were no carry in column 3 then $E = N$, which is impossible. Therefore there is a carry and $N = E + 1$. If there were no carry in column 2, then $(N + R) \,(\mathrm{mod} 10) = E$, and $N = E + 1$, so $(E + 1 + R) \pmod{10} = E$ which means $(1 + R) \pmod{10} = 0$, so $R = 9$. But $S = 9$, so there must be a carry in column 2, and so $R = 8$. To produce a carry in column 2, we must have $D+E = 10+Y$. Now Y is at least 2 so $D+E$ is at least 12. The only two pairs of available numbers that sum to at least 12 are (5,7) and (6,7) so either $E = 7$ or $D = 7$. Since $N = E + 1$, E can't be 7 because then $N = 8 = R$ so $D = 7$. Finally E can't be 6 because then $N = 7 = D$ so $E = 5$ and $D + E = 12$ so $Y = 2$. Our final sum is therefore $9,567 + 1,085 = 10,652$.

Comment This puzzle type goes under the heading of *verbal puzzles* or *alphametics*.

Problem Set 4

1.

$$(3675, 2058) \mapsto (2058, 1617) \mapsto (1617, 441) \mapsto (1176, 441)$$
$$\mapsto (735, 441) \mapsto (441, 294) \mapsto (294, 147) \mapsto (147, 147).$$

Hence the hcf of 3675 and 2058 is 147.

2.

$$516 = 1 \times \underline{432} + \underline{84}$$
$$432 = 5 \times \underline{84} + \underline{12}$$
$$84 = 7 \times \underline{12};$$

Hence the hcf of 516 and 432 is 12.

3. Starting from the penultimate line of the calculation:

$$12 = 432 - (5 \times 84) \Rightarrow$$
$$84 = 516 - 432 \Rightarrow$$

$$12 = 432 - 5(516 - 432) = 432 - (5 \times 516) + (5 \times 432) \Rightarrow$$

$$12 = 6 \times 432 - 5 \times 516.$$

Hence $m = -5$ and $n = 6$.

Comment This process of reversing the algorithm, beginning with the penultimate equation and working each equations in reverse, eliminating the intermediate remainders at each stage, will always yield the gcd expressed as a linear combination of the original number pair. In this way the hcf of a pair of numbers can be expressed as a linear combination of the two numbers in question and this is very important for both practical and theoretical reasons. It forms the basis for the general method of solving linear congruences, that is equations of the form $ax \equiv b \,(\bmod\; m)$(meaning find x such that m is a factor of $ax - b$) and more generally for solving systems of such congruences using what is known as the *Chinese Remainder Theorem*, that name arising because this problem type was popular in ancient Chinese problem sets. Moreover, if two numbers a and b are *coprime* , meaning that their hcf is 1, then the algorithm can be reversed to find integers (of opposite signs) x and y such that $ax + by = 1$. This fact is often exploited in number theory including in the proof of *Euclid's Lemma,* which says that if p is a prime factor of a product ab, then p divides at least one of the numbers a and b. The modern theory of internet cryptography is very firmly based on a body results which all stem from the euclidean algorithm.

4.

$$35 = 22 + 13$$

$$22 = 13 + 9$$

$$13 = 9 + 4$$

$$9 = 2 \times 4 + 1.$$

Working these equations backwards then gives

$$1 = 9 - 2(4) = 9 - 2(13 - 9) = -2(13) + 3(9)$$

$$= -2(13) + 3(22 - 13) = 3(22) - 5(13) = 3(22) - 5(35 - 22)$$

$$= 8(22) - 5(35)$$

and so $m = 8$ and $n = -5$.

5. We have the lcm of 9 and 15 is 45 so we want $(\frac{20}{45}, \frac{24}{45}) = \frac{(20,24)}{45} = \frac{4}{45}$.

6. We want the least integers m and n such that $\frac{4m}{9} = \frac{8n}{15}$, which gives $\frac{m}{n} = \frac{8\times 9}{15\times 4} = \frac{72}{60} = \frac{6}{5}$. Hence the least common multiple is $\frac{4}{9} \times 6 = \frac{8}{15} \times 5 = \frac{24}{9} = \frac{8}{3}$.

7. Let us begin by denoting $\frac{(a,c)}{[b,d]}$ by h, where $[b,d]$ denotes the lcm of b and d. Write $r = \frac{a}{b}$ and $s = \dfrac{c}{d}$. Then

$$\frac{r}{h} = \frac{a}{b} \cdot \frac{[b,d]}{(a,c)} = \frac{a}{(a,c)} \cdot \frac{[b,d]}{b}$$

and by definition of highest common factor and least common multiple, both numbers in the preceding product are positive integers, and so $h|r$ (meaing h is a factor of r). By the same argument, $h|s$ also and hence h is a common factor of r and s.

Conversely, let $y = \frac{e}{f}$ be any common factor of r and s so that

$$\frac{a}{b} = k\frac{e}{f}, \ \frac{c}{d} = l\frac{e}{f} \text{ say with } k, l \in \mathbb{Z}^+.$$

Without loss we may now take $(a,b) = (c,d) = (e,f) = 1$. Since $(a,b) = 1$ there exists $t \in \mathbb{Z}^+$ such that $ke = ta$ and $f = tb$. It follows that f is a multiple of b and, by the same token, a multiple of d and so $f = u[b,d]$ for some $u \in \mathbb{Z}^+$. Next we verify that $e|a$. If this were not so that there would exist a prime factor p of e such that $p|t$. But then $p|t||f$ so that p is a common factor of e and f, contrary to the condition that $(e,f) = 1$. Hence $e|a$ and likewise we have that $e|c$ so that $e|(a,c)$. Therefore we may write $(a,c) = ve$ for some $v \in \mathbb{Z}^+$ giving us:

$$\frac{e}{f} = \frac{(a,c)}{vu[b,d]} \leq \frac{(a,c)}{[b,d]} = h,$$

which shows that h is indeed the highest common factor of r and s. Finally we note that it follows from the previous equation that h is a multiple (vu) of any common factor $\frac{e}{f}$ of r and s.

8. For positive rational numbers $r = \frac{a}{b}$ and $s = \frac{c}{d}$ the least common multiple l exists and

$$l = [\frac{a}{b}, \frac{c}{d}] = \frac{[a,c]}{(b,d)}.$$

For the given number l is a multiple of r as

$$\frac{l}{r} = \frac{[a,c]}{(b,d)} \cdot \frac{b}{a} = \frac{[a,c]}{a} \cdot \frac{b}{(b,d)}$$

and both terms in the product in are integers. Similarly $\frac{l}{s} \in \mathbb{Z}^+$ and so l as defined in is a multiple of both r and s.

Next suppose y is a multiple of r and s so that $y = \frac{e}{f} = \frac{ka}{b} = \frac{mc}{d}$ for some $k, m \in \mathbb{Z}^+$. Without loss we may now take $(a,b) = (c,d) == (e,f) = 1$. Since $(e,f) = 1$ there exists $t \in \mathbb{Z}^+$such that $te = ka$ and $tf = b$. Hence $f|b$ and by the same token $f|d$ so that $f|(b,d)$ and we may write $(b,d) = uf$ say. Next we show that $a|e$. If we suppose this is not the case, it would follow from $te = ka$

that there exists a prime p such that $p|a$ and $p|t$. It then follows from $tf = b$ that $p|b$ also, contradicting that $(a, b) = 1$. Hence $a|e$ and likewise we may infer that $c|e$ also. Hence e is a multiple of both a and c and so $e = v[a, c]$ for some $v \in \mathbb{Z}^+$. From this analysis we conclude that

$$\frac{e}{f} = uv\frac{[a, c\}}{(b, d)} = uvl$$

so that l is indeed the least common multiple of r and s and $l|y$ as claimed.

9. Assume that $(a, b) = (c, d) = 1$. Let p be a prime factor of (a, c) so that $p|a$ and $p|c$. If it were the case that $p|[b, d]$ then $p|b$ or $p|d$. In the first case it would then follows that $p|(a, b)$ and in the second that $p|(c, d)$, giving a contradicting at least one of the assumptions that $(a, b) = 1$ and $(c, d) = 1$. Therefore h as given in will be in reduced form if the same is true of the representations of r and s.

Next suppose that the given fractions are in reduced form and let p be a prime factor of (b, d) so that $p|b$ and $p|d$. If p were also a factor of $[a, c]$ then p would also be a factor of at least one of a and c giving the contradiction that at least one of the pairs (a, b) and (c, d) were not coprime. Hence we conclude that there is no such common factor p and that l as given is in reduced form.

10. $\frac{8}{15} - \frac{4}{9} = \frac{24-20}{45} = \frac{4}{45}$. $\frac{4}{9} - \frac{4}{45} = \frac{20-4}{45} = \frac{16}{45}$. $(\frac{16}{45}, \frac{4}{45}) = \frac{4}{45}$. Hence $(\frac{4}{9}, \frac{8}{15}) = \frac{4}{45}$.

Problem Set 5

1. $323 = 17 \times 19$.

2. The first failure occurs for $n = 20$: $6 \times 20 - 1 = 119 = 7 \times 17$ and $6 \times 20 + 1 = 121 = 11^2$, so neither of $6n \pm 1$ is prime for $n = 20$.

3. Every integer has exactly one of the six forms $6n - 2, 6n - 1, 6n, 6n + 1, 6n + 2, 6n + 3$. We note that $6n \pm 2$ is even while $6n$ and $6n + 3$ are divisible by 3. Hence, with the exceptions of 2 and 3, every prime number has the form $6n \pm 1$ for some n.

4. The next pair of consecutive twin primes is $(101, 103)$ and $(107, 109)$.

5. Put $n = a$ and we get $a + ad = a(1 + d)$ and since both factors are at least 2, it follows that this number is not prime.

6. This time put $n = d + 2$ we get $1 + d(d + 2) = d^2 + 2d + 1 = (d + 1)^2$, which is composite.

7. Smallest example is the obvious one of $x = 41$ for then we get $41^2 - 41 + 41 = 41^2$.

8. Following the hint, consider the 100 consecutive integers $(101)!+2$, $(101)!+3, \cdots, (101)! + 101$. Each of these numbers is composite since for $2 \leq k \leq 101$ we have $k|(101)! + k$.

9. We only need that $p = 2n + 1$ is odd as then

$$p^2 - 1 = (2n + 1)^2 - 1 = 4n^2 + 4n = 4n(n + 1);$$

since exactly one of the factors n and $n + 1$ is even, we see that $p^2 - 1$ has a factor of $4 \times 2 = 8$.

10. Now $p^4 - 1 = (p^2 - 1)(p^2 + 1) = (p - 1)(p + 1)(p^2 + 1)$. Since $p > 5$ and p is prime it follows that p is not divisible by either 2, 3 nor 5. Since p is odd, each of $p-1$, $p+1$ and p^2+1 is even. Indeed exactly one of $p-1$ and $p+1$ is a multiple of 4 so that $2\times4\times2 = 16|p^4-1$. Similarly if $p \equiv 1 \pmod 3$ then $3|p-1$, while if $p \equiv -1 \pmod 3$ then $3|p + 1$ so that $3|p^4 - 1$ also. Finally $p \equiv 2$ or $-2 \pmod 5$ so $p^2+1 \equiv 0 \pmod 5$ whence $5|p^4-1$. Putting these products together we infer that $16 \times 3 \times 5 = 240$ is a factor of $p^4 - 1$. To see that 240 is the greatest common factor of our set we note that $7^4 - 1 = 2400 = 240 \times 10$ while $11^4 - 1 = 14640 = 240 \times 61$ and so in the case of the primes $p = 7$ and $q = 11$ the greatest common factor of the number pair $p^4 - 1$ and $q^4 - 1$ is indeed 240.

Problem Set 6

1. Alexander must be innocent for if he were lying Barbara would not accuse Caroline. It follows that Alexander is telling the truth and Barbara broke the window. (That accounts for her lie and is consistent with both Caroline and David not knowing what happened.)

Comment Logic puzzles are quite fun and lead to a lot of interesting questions in analysis of mathematical trees. Look up *The Land of Knights and Knaves* to learn more.

2. At each snap, the number of pieces increases by 1. Since we begin with one block and end with n squares, the number of snaps is therefore $n - 1$.

Comment This shows that the outcome is independent of the way we go about it so that $n - 1$ is not only the minimum but the maximum number of snaps to complete the task.

3. Since the claim of each octopus contradicts all of the others, there must be at least 3 liars. If all 4 were liars, they would each have 7 legs, giving $4 \times 7 = 28$ legs in all but then the golden octopus would be telling the truth, which is inconsistent. Hence there are 3 lying octopi and one truthful octopus. The 3 liars then have $3 \times 7 = 21$ legs between them. The truthful octopus then has 6 or 8 legs. If he had 8 legs then there would be $21 + 8 = 29$ legs overall, and all would be lying, which is inconsistent. Hence the truthful octopus has 6 legs and there are $21 + 6 = 27$ legs overall, which is the claim of the green octopus. Therefore the blue, red and golden octopi each have 7 legs while the green one has 6.

4. There are two solutions but both involve seven crossings in all. He takes the Goose across, goes back and picks up either the Bag or the Dog, takes that across, picks up Goose and takes her back, then picks up the item he left behind and takes that across, and then goes back to pick up the Goose. The key thing is to keep the Goose away from the other two.

5. Let use write (a, b, c) to denote the amount of wine in each of pitchers from smallest to largest, so they start with the triple $(0, 0, 8)$. All they can do is pour wine from one container to a second until either the first is empty or the second is full. They can then reach the desired state of $(0, 4, 4)$ in seven steps beginning with $(0, 0, 8) \to \cdots$ as follows:

$$(0, 5, 3) \to (3, 2, 3) \to (0, 2, 6) \to (2, 0, 6) \to (2, 5, 1) \to (3, 4, 1) \to (0, 4, 4).$$

6. Yes, for there are two possibilities. If Anne is married then, since Anne is looking at the bachelor George, a married person is looking at an unmarried one. Alternatively, Anne is unmarried and the married Jack is looking at the unmarried Anne. One of these two possibilities must apply so we can conclude that a married person is looking at an unmarried one.

7. On each turn, I drink twice as much as you so that the ratio of our shares is $2 : 1$. Hence I drink $\frac{2}{2+1} = \frac{2}{3}$ of the pint of juice. Or setting up the infinite GP shows my share to be:

$$\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{16} + \cdots + \frac{1}{2} \cdot \frac{1}{4^{n-1}} + \cdots = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{4}} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}.$$

8. I was in the middle and below 15th place. If my position were 13th or below, there would be at most 12 below and 12 above me, making at most 25 students in all, when we know there were at least 26. Therefore I must have been between 13th and 15th place, which is to say I was 14th in line. I was therefore standing next to the object of my dreams, with 13 class mates flanking me on either side, making 27 of us in total.

9. At each point in the journey the best you can do is to choose the tangent line to the circle with common centre to the main circle and whose radius is

the line from the centre to the point where you are standing. In this case, the direction chosen by the tyrant will not affect your final distance from the centre after you take your step. In you choose any other line, the tyrant can choose a direction which will make you relatively worse off. Hence this is an optimum strategy.

Given that you adopt this approach, we show by induction that your distance from the centre after step k is $\sqrt{k}$, which is clearly true for your first step. Assuming inductively that this holds after k steps, by Pythagoras, your distance d from the centre after $k+1$ steps will be given by $d^2 = 1^2 + (\sqrt{k})^2 = 1 + k$ so that your new distance is indeed $d = \sqrt{k+1}$ and the induction continues. To escape the circle will therefore take k steps where $\sqrt{k} = n$, which is to say that $k = n^2$ steps will be needed.

10. The total number of games is $(10+15+17)/2 = 21$ of which, B plays C $21 - 10 = 11$times. Since A and C cannot play consecutive games against each other, this is only possible if they play all the odd numbered games $1, 3, \cdots, 21$, which means that A plays all the even numbered games and loses them all. Hence A lost the second game.

Problem Set 7

1.
$$(A \cap B) \cup (B' \cap A) = (A \cap B) \cup (A \cap B') =$$
$$A \cap (B \cup B') = A \cap \mathcal{U} = A,$$

where $\mathcal{U}$ denotes the universal set. The dual statement (which must also hold) is

$$(A \cup B) \cap (B' \cup A) = A.$$

2. Using De Morgan's Law

$$(A' \cap B)' \cap (A \cup B) = \big((A')' \cup B'\big) \cap (A \cup B) =$$
$$(A \cup B') \cap (A \cup B) = A \cup (B' \cap B) = A \cup \emptyset = A.$$

Comment Set Laws involving only the so called boolean set operations of $\cap, \cup, '$ come in dual pairs (with $\emptyset$ and $\mathcal{U}$ interchanging), the pair known as De Morgan Laws being $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$.

3. The number of subsets of a set of size n is 2^n. The number with no more than 1 member is $\binom{n}{0} + \binom{n}{1}$. Applying these observations here gives as the answer:

$$2^{12} - (1 + 12) = 4,096 - 13 = 4,083.$$

4. Writing F and G for the set of students taking French and German respectively, we require the number $|(F \cup L)'| = 100 - |F \cup L|$. Now $|F \cup L| = |F| + |L| - |F \cap L| = 50 + 40 - 20 = 70$. Therefore the number of students not taking either language is $100 - 70 = 30$.

Comment This *Counting Principle* can be extended to three or more sets - the rule that develops is sometimes called the *Inclusion-Exclusion Principle* (see MA202 Set 9) and it gives a sum, alternating in sign, which counts the number of members of a finite union of finite sets. For three sets it has the form:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

and from here the general pattern of the result is not hard to see.

5. For $n \geq 3$, treat the union inside the brackets as that of two sets and apply the $n = 2$ case:

$$A \cap ((B_1 \cup B_2 \cup \cdots \cup B_{n-1}) \cup B_n) = A \cap (B_1 \cup B_2 \cup \cdots \cup B_{n-1}) \cup (A \cap B_n).$$

Next apply the inductive hypothesis to the first bracket to get the required expression:

$$(A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_{n-1}) \cup (A \cap B_n).$$

Comment: There is of course a dual law to this one that comes from interchanging $\cap$ and $\cup$ throughout, which is equally true. It is a common feature of mathematics texts that a formula of some kind is established for two objects and then the comment is made that the formula immediately extends to $n \geq 2$ objects. Implicit in any such claim is that the case for $n \geq 3$ objects can be reduced at once to the $n = 2$ case by this type of inductive argument.

6.

p	$\sim p$	$p \vee (\sim p)$
T	F	T
F	T	T

Since the right hand column consists entirely of T's, $p \vee (\sim p)$ is a tautology.

7.

p	q	r	$q \vee r$	$p \wedge q$	$p \wedge r$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
F	T	T	T	F	F	F	F
T	F	F	F	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Since the final two columns are identical, it follows that $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

8.

p	q	$\sim p$	$\sim q$	$p \to q$	$(\sim q) \to (\sim p)$	$\sim (p \wedge (\sim q))$
T	T	F	F	T	T	T
T	F	F	T	F	F	F
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Since the final three columns agree, we infer that $p \to q \equiv (\sim q) \to (\sim p) \equiv \sim (p \wedge (\sim q))$.

9.

p	q	$p \to q$	$q \to p$	$(p \to q) \equiv (q \to p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Since there is an F in the final column, the statement at the top of that column is not true.

10.

p	q	r	$p \vee q$	$p \to r$	$q \to r$	r
T	T	T	T	T	T	T*
T	T	F	T	F	F	F
T	F	T	T	T	T	T*
F	T	T	T	T	T	T*
T	F	F	T	F	T	F
F	T	F	T	T	F	F
F	F	T	F	T	T	T
F	F	F	F	T	T	F

The conclusion column (far right) has T entries in each of the three critical rows (rows in which all the premises have the value T) and so $p \vee q$, $p \to r$, $q \to r$, $\therefore r$ is a valid argument.

Problem Set 8

1. For $n = 1$ both sides of the expression return the value 2, thereby grounding the induction. Suppose that $n \geq 2$. By the inductive hypothesis the LHS of the expression take the form:

$$\sum_{k=1}^{n-1} k(k+1) + n(n+1) = \frac{(n-1)n(n+1)}{3} + n(n+1)$$

$$= \frac{n^2(n+1) - n(n+1) + 3n(n+1)}{3} = \frac{n(n+1)(n+2)}{3},$$

thus completing the inductive verification of the identity.

2. For the second equality note that:

$$\frac{1}{4}n^2(n+1)^2 = \Big(\frac{1}{2}n(n+1)\Big)^2 = (1 + 2 + \cdots + n)^2.$$

As for the first equality, for $n = 1$ both sides output the number 1, thus anchoring the induction. Now we suppose that $n \geq 2$ and apply the inductive hypothesis to the RHS to get:

$$1^3 + 2^3 + \cdots + (n-1)^3 + n^3 = \frac{1}{4}(n-1)^2 n^2 + n^3$$

$$= \frac{1}{4}n^2\big((n^2 - 2n + 1) + 4n\big) = \frac{1}{4}n^2(n^2 + 2n + 1) = \frac{1}{4}n^2(n + 1)^2.$$

Comment: note again that the proof was easy but we did need the formula beforehand. A general approach to establishing these formulae recursively was outlined in Set 9 of MA101, Problems 7 & 8. A general formula for sum of powers that expresses the sum of kth powers as a polynomial of degree $k+1$ dates back to the German mathematician Johann Faulhaber (1580-1635). These polynomials involve so called *Bernoulli numbers* in the coefficients, which arise in many problems of combinatorics and repeated differentiation of common functions, such as tan and sec.

3. For $n = 1$ we get $n(n^2 + 5) = 6$ is a mutliple of 6. Suppose the result holds for $k = n$ and consder the next case:

$$(n+1)((n+1)^2+5) = (n+1)(n^2+2n+6) = n(n^2+5)+n(2n+1)+n^2+2n+6$$

$$= n(n^2 + 5) + (3n^2 + 3n + 6) = n(n^2 + 5) + 3(n(n + 1) + 2);$$

now the first term is a multiple of 6 by the inductive hypothesis, and since $n(n+1)$ is even (as exactly one of the factors is even) it follows that $3(n(n+1)+2)$ is also a multiple of 6. Therefore $(n + 1)((n + 1)^2 + 5)$ is a multiple of 6 and the induction continues.

Comment Aternatively, we might checked that the difference $d = (n+1)((n+1)^2 + 5) - n(n^2 + 5)$ is divisible by 6, whereupon the result follows by induction.

4. Both sides of the formula agree for $n = 1$ so let us take $n \geq 2$, whence, using the inductive hypothesis the LHS becomes:

$$a \cdot \frac{1 - r^{n-1}}{1 - r} + ar^{n-1} = a\Big(\frac{(1 - r^{n-1}) + r^{n-1}(1 - r)}{1 - r}\Big)$$

$$= a \cdot \frac{1 - r^{n-1} + r^{n-1} - r^n}{1 - r} = a \cdot \frac{1 - r^n}{1 - r},$$

and the induction continues, thus establishing the result.

Comment A more natural way perhaps to prove the summation formula for a geometric progression is to multiply both sides by $1 - r$ and note that when we expand the LHS, which now has the form:

$$a(1 - r)(1 + r + r^2 + \cdots + r^{n-1}),$$

the expression 'telescopes' (meaning that all but a fixed number of terms cancel) to give $a(1 - r^n)$, in agreement with the RHS. The disadvantage in proving a formula by induction is that the formula first needs to be identified - guessed if you like - before it can be proved. It is therefore good if we can discover a way that we may travel from one side of the formula to the other by some general algebraic method. Nonetheless many, many results in mathematics are established by inductive argument.

5. From the recursion we obtain the Fibonacci series as far as F_{10} as $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots$.

6. For $n = 0$ the assertion says that $F_0 = F_2 - 1$, which is to say $0 = 1-1$,which is true. Now let $n \geq 1$. Using induction and the Fibonacci recursion gives:

$$\sum_{k=0}^{n} F_k = \sum_{k=0}^{n-1} F_k + F_n = (F_{n+1} - 1) + F_n =$$

$$(F_n + F_{n+1}) - 1 = F_{n+2} - 1,$$

as required for the induction to continue and so establish the result.

7.

$$0, 1, -1, 2, -3, 5, -8, 13, -21, 34, \cdots .$$

8. It would seem that $f_n = (-1)^n F_n$, $n = 0, 1, 2, \cdots$, where f_n denotes our sequence and the F_n the normal Fibonacci numbers. By Question 5, we see that the formula works for $n = 0$ and $n = 1$ so let us assume that it is valid for all values up to some $n \geq 1$ and consider f_{n+1}. By definition of the sequence we have

$$f_{n-1} = f_n + f_{n+1} \Rightarrow f_{n+1} = f_{n-1} - f_n$$

$$\Rightarrow f_{n+1} = (-1)^{n-1} F_{n-1} - (-1)^n F_n = (-1)^{n+1} F_{n-1} + (-1)^{n+1} F_n$$

$$= (-1)^{n+1}(F_{n-1} + F_n) = (-1)^{n+1} F_{n+1}, \text{ as required.}$$

9.

$$a_0 = 0,\ a_1 = 1,\ a_2 = \frac{0+1}{2} = \frac{1}{2},\ a_3 = \frac{1}{2}\big(1 + \frac{1}{2}\big) = \frac{3}{4},$$

$$a_4 = \frac{1}{2}\big(\frac{1}{2} + \frac{3}{4}\big) = \frac{5}{8},\ a_5 = \frac{1}{2}\big(\frac{3}{4} + \frac{5}{8}\big) = \frac{11}{16},$$

$$a_6 = \frac{1}{2}\big(\frac{5}{8} + \frac{11}{16}\big) = \frac{21}{32},\ a_7 = \frac{1}{2}\big(\frac{11}{16} + \frac{21}{32}\big) = \frac{43}{64},\ a_8 = \frac{1}{2}\big(\frac{21}{32} + \frac{43}{64}\big) = \frac{85}{128}.$$

10. The *characteristic equation* here is $2x^2 - x - 1 = (2x + 1)(x - 1) = 0$, which has roots 1 and $-\frac{1}{2}$. The general solution is then $a_n = A + (-1)^n B/2^n$. Applying the initial conditions that $a_0 = 0$ and $a_1 = 1$ gives the equations $A + B = 0$ and $A - \frac{1}{2}B = 1$. Hence $A = \frac{2}{3}$ and $B = -\frac{2}{3}$. The solution is then

$$a_n = \frac{2}{3}\Big(1 + \frac{(-1)^{n+1}}{2^n}\Big),\ n = 0, 1, 2, \cdots .$$

Letting n increase without bound then gives $\lim_{n\to\infty} a_n = \frac{2}{3}$.

Problem Set 9

1.
$$\frac{17!18!}{2} = (17!)^2 \cdot \frac{18}{2} = (17!)^2 \cdot 9 = (3 \times 17!)^2.$$

2. The number of boys currently is $\frac{3}{3+5} \cdot 48 = \frac{3}{8} \cdot 48 = 18$ and so there are $(48 - 18) = 30$ girls. In the enlarged class of x pupils theration of girls to boys is $3 : 5$ so we have $\frac{3}{8} \cdot x = 30$ and hence $x = \frac{8}{3} \cdot 30 = 80$. The number of boys in the bigger class is thus $80 - 30 = 50$ and so we would need $50 - 18 = 32$ additional boys to achieve this.

3. Let $a = \sqrt{3\sqrt{2\sqrt{3\sqrt{2\cdots}}}}$. Then by squaring and squaring again we get

$$(a^2)^2 = 3^2 \cdot 2\sqrt{3\sqrt{2\sqrt{3\sqrt{2\cdots}}}} \Rightarrow a^4 = 18a \Rightarrow a^3 = 18;$$

or if you prefer a is the cube root of 18.

4. Calling the number in question x we see that

$$(x^2 - 4)^2 = 4 - x \Rightarrow x^4 - 8x^2 + x + 12 = 0$$
$$\Rightarrow (x^2 - x - 3)(x^2 + x - 4) = 0;$$

clearly $x > 2$ so the roots of the second factor are not relevant. We take the postive root of the first factor, which is

$$x = \frac{1 + \sqrt{13}}{2} \approx 2 \cdot 3028.$$

5. We have, working modulo 4 throughtout that $0^2 \equiv 0$, $1^2 \equiv 1$, $2^2 \equiv 0$ and $3^2 \equiv 0$. Hence the sum of two squares takes on one of the values $0 + 0 \equiv 0$, $0 + 1 \equiv 1$, $1 + 1 \equiv 2$, so $a^2 + b^2 \equiv 3$ (modulo 4) never arises.

6. Working modulo 8 a square may take on the values $0^2 \equiv 0, 1^2 \equiv 1$, $2^2 \equiv 4$, $3^2 \equiv 1$, $4^2 \equiv 0$, $(-3)^2 \equiv 1$, $(-2)^2 \equiv 4$, or $(-1)^2 \equiv 1$. Hence the sum of three squares is equal to the sum of three numbers, with repeats allowed, from the set $\{0, 1, 4\}$. Clearly a sum of 7 is not possible (although all other values do arise). Therefore the sum of three squares cannot equal 7 modulo 8, which is to say cannot have the form $8k + 7$.

7. In general $a^2 \equiv 0$ or 1 (mod 4). If we had a solution to the equation in which $x^2 \equiv 0$ (mod 4) then, working modulo 4,

$$19y^2 - 15 \equiv 3y^2 - 1 \equiv 0 \Rightarrow 3y^2 \equiv 9 \Rightarrow y^2 \equiv 3 \pmod 4$$

and so this is a contradiction. The alternative is that $x^2 \equiv 1$ in which case

$$19y^2 - 15 \equiv 3y^2 - 1 \equiv 1 \Rightarrow 3y^2 \equiv 6 \Rightarrow y^2 \equiv 2 \pmod 4,$$

which is also a contradiction. Hence there are no lattice points on the curve defined by $x^2 - 19y^2 = 15$.

8. $\{1, 2, 4, 7, 8, 11, 13, 14\}$, so that $\phi(15) = 8$; or by the formula, since the prime divisors of 15 and 3 and 5:

$$\phi(15) = 15(1 - \frac{1}{3})(1 - \frac{1}{5}) = 15(\frac{2}{3})(\frac{4}{5}) = 8.$$

For $323 = 17 \times 19$ we get

$$\phi(323) = 323(1 - \frac{1}{17})(1 - \frac{1}{19}) = 323 \cdot \frac{16}{17} \cdot \frac{18}{19} = 16 \cdot 18 = 288.$$

9. Observe that the terms in the sum are the difference between successive kth roots of unity. Hence, for a given n, the sum represents the length of the boundary of the regular n-gon inside the unit circle, the limit of which is the circumference of that circle, which is 2π.

10. No, because in general if $a \geq 2$ and n has an odd factor m so that $n = mt$ then we have the following factorization:

$$a^n + 1 = a^{mt} + 1 = (a^t + 1)(a^{(m-1)t} - a^{(m-2)t} + a^{(m-3)t} - \cdots + 1).$$

In this instance $a = 2$ and $n = 50 = 5 \times 10$ so that we may take $m = 5$ and $t = 10$. The preceding factorization yields:

$$2^{50} + 1 = (2^{10} + 1)(2^{40} - 2^{30} + 2^{20} - 2^{10} + 1).$$

Comment Clearly if a is odd then $a^n + 1$ is even and so not prime. It follows that the only numbers of the form $a^n + 1$ that can be prime have the form $2^{2^n} + 1$. These are called the *Fermat numbers*, some of which are primes, and primes with very special properties.

Problem Set 10

1. Following the rules we get $7 \mapsto 22 \mapsto 11 \mapsto 34 \mapsto 17 \mapsto 52 \mapsto 26 \mapsto 13 \mapsto 40 \mapsto 20 \mapsto 10 \mapsto 5 \mapsto 16 \mapsto 8 \mapsto 4 \mapsto 2 \mapsto 1$.

Comment A long standing conjecture is that, beginning with *any* number, this process always ends in 1. The term 'hailstone numbers' comes about because of the pattern of the number sequences so generated, which generally show a number of erratic rises and falls yet eventually always hit the ground. Or so it seems!

2. $276 = 2^2 \times 3 \times 23$. Hence

$$a(276) = \frac{2^3-1}{2-1}\cdot\frac{3^2-1}{3-1}\cdot\frac{23^2-1}{23-1} = 7\times\frac{8}{2}\times\frac{528}{22} = 672.$$

Comment We see that 276 is an *abundant* number: a number that is less than the sum of its factors that are less than the number itself. Curiously the sequence $a(276), a^2(276), a^3(276), \cdots$ is unknown in that it may be, as far as anyone knows, an infinite non-repeating sequence!

3. By definition we have $S(n,1) = 1$. The number of ways of partitioning the n-set into two blocks, listed in a specific order, is 2^n as there are two choices of box in which to place each member of the set. Since the sets are not in any particular order, the number of such pairs of sets is $\frac{2^n}{2} = 2^{n-1}$. Finally the case where one of the sets is empty must be excluded, and so $S(n,2) = 2^{n-1} - 1$.

4. Again by definition we have $S(n,n) = 1$. Next, a partition of an n-set into $n-1$ non-empty block is determined by the choice of the unique block with 2 members, (the remaining blocks being singleton sets). Hence $S(n, n-1) = \binom{n}{2} = \frac{1}{2}n(n-1)$.

Comment Note that, in contrast to the binomial coefficients, we do *not* have row symmetry for these Stirling numbers in that $S(n,r) \neq S(n, n-r)$. However it follows from the definition that the $S(n,r)$ satisfy the recurrence $S(n,r) = S(n-1, r-1) + rS(n-1, r)$, which is the same as that for the binomial coefficients except for the introduction of the multiplier of r in the second term. *Stirling Numbers of the First Kind* are related but count something quite different, that being the number of ways we can permute n objects into r cycles.

5.

$$\frac{7!!}{7!} = (7!-1)! = x!$$

and so $x = 7! - 1 = 5040 - 1 = 5039$.

6.

$$\frac{45}{16} = 2 + \frac{13}{16} = 2 + \frac{1}{1+\frac{3}{13}} = 2 + \frac{1}{1+\frac{1}{4+\frac{1}{13}}}$$

so that $\frac{45}{16} = [2; 1.4.13]$.

7.

$$\frac{16}{45} = \frac{1}{2+\frac{13}{16}} = \frac{1}{2+\frac{1}{1+\frac{3}{13}}} = \frac{1}{2+\frac{1}{1+\frac{1}{4+\frac{1}{13}}}}$$

so that $\frac{16}{45} = [0; 2, 1, 4, 13]$.

Comment: Quite generally, a positive rational number $\frac{p}{q} < 1$ has the same expansion as that as $\frac{q}{p}$, just shifted one place to the right.

8.$\sqrt{2} = 1 + (\sqrt{2} - 1)$ so that $a_0 = 1$. Then

$$r_1 = \frac{1}{r_0 - a_0} = \frac{1}{\sqrt{2} - 1} = \frac{1 + \sqrt{2}}{(\sqrt{2} - 1)(\sqrt{2} + 1)} = 1 + \sqrt{2} \Rightarrow a_1 = \lfloor 1 + \sqrt{2} \rfloor = 2;$$

$$r_2 = \frac{1}{r_1 - a_1} = \frac{1}{(1 + \sqrt{2}) - 2} = \frac{1}{\sqrt{2} - 1} = r_1.$$

Hence $r_1 = r_2 = \cdots$, $a_1 = a_2 = \cdots = 2$ and so $\sqrt{2} = [1; 2, 2, 2, \cdots]$, which is written in recurring notation as $[1; \overline{2}]$.

9. $\sqrt{7} = 2 + (\sqrt{7} - 2)$ so that $a_0 = 2$. Then

$$r_1 = \frac{1}{\sqrt{7} - 2} = \frac{\sqrt{7} + 2}{7 - 4} = \frac{1}{3}(2 + \sqrt{7}) \Rightarrow a_1 = \lfloor r_1 \rfloor = 1;$$

$$r_2 = \frac{1}{r_1 - a_1} = \frac{3}{\sqrt{7} - 1} = \frac{3\sqrt{7} + 3}{6} = \frac{\sqrt{7} + 1}{2} \Rightarrow a_2 = \lfloor r_2 \rfloor = 1$$

$$r_3 = \frac{1}{r_2 - a_2} = \frac{2}{\sqrt{7} - 1} = \frac{2\sqrt{7} + 2}{6} = \frac{\sqrt{7} + 1}{3} \Rightarrow a_3 = \lfloor r_3 \rfloor = 1$$

$$r_4 = \frac{1}{r_3 - a_3} = \frac{3}{\sqrt{7} - 2} \cdot \frac{\sqrt{7} + 2}{\sqrt{7} + 2} = \frac{3\sqrt{7} + 6}{3} = \sqrt{7} + 2 \Rightarrow a_4 = \lfloor r_4 \rfloor = 4$$

$$r_5 = \frac{1}{r_4 - a_4} = \frac{1}{\sqrt{7} - 2} = r_2.$$

Hence $\sqrt{7} = [2; \overline{1, 1, 1, 4}]$.

10. $\lfloor \phi \rfloor = 1$ so that $a_0 = 1$. Then

$$r_1 = \frac{1}{\phi - 1} = \frac{2}{\sqrt{5} - 1} = \frac{2\sqrt{5} + 2}{4} = \frac{1 + \sqrt{5}}{2} = \phi = r_0.$$

Hence $\phi = [1, \overline{1}]$.

Comment Truncation of continued fraction expansions generally give the best rational approximations to irrational numbers for a given size of denominator. Since the expansion of the Golden ratio consists entirely of 1's this convergence is as slow as possible, so that ϕ is among the most difficult of numbers to approximate accurately by rational numbers. This solution can be found by reversing the expansion, as you have already seen is Set 3 Question 4.