Mathematics 301 Algebraic semigroups

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Problem Set 1 Elementary properties and examples

A *semigroup* $(S, \circ)$ is a set S with an *associative binary operation* $\circ$, which is often denoted by juxtaposition so that $x(yz) = (xy)z$ for all $x, y, z \in S$. If S has an identity element 1, then S is a *monoid*. For a semigroup S that is not a monoid, the monoid $S^1 = S \cup \{1\}$ is the semigroup S with 1 the *adjoined identity element*; if S is a monoid then we take $S^1 = S$. The set of idempotents of S (which may be empty), is denoted by $E(S)$. We write $S \leq T$ to denote that S is a subsemigroup of the semigroup T.

For one-sided definitions, we often record a one-sided version only, the alternate notion being then implicitly defined using left-right symmetry.

1. A semigroup is *left cancellative* if $ax = ay$ implies $x = y$ $(a, x, y \in S)$.

(a) Show that every idempotent in a left cancellative semigroup is a left identity element.

(b) Show that a *cancellative* (i.e. left and right cancellative) semigroup S can have at most one idempotent e which is then the identity element of S.

2. A semigroup S is *left simple* if $Sa = S$ for all $a \in S$. Prove that a semigroup S is a group if and only if S is both left and right simple.

3. A *right ideal* $I \neq \emptyset$ of a semigroup S is a subset of S such that $IS \subseteq S$; we say that I is an *ideal* of S if I is both a left and a right ideal of S.

(a) Show that the smallest right ideal I containing a non-empty subset A of S is $I = AS^1$.

(b) Similarly the ideal of I of S *generated by* A is $I = S^1AS^1$.

Comment If $A = \{a\}$ we speak of the *principal right ideal* aS^1 and *principal ideal* S^1aS^1.

4. Let X be a set and define $\mathcal{T}_X$ as the semigroup of all mappings on X under function composition (composed from left to right).

(a) Show that $\mathcal{T}_X$ is a monoid that contains the symmetric group $\mathcal{S}_X$.

(b) Show that the set C of constant mappings in $\mathcal{T}_X$ form a *right zero semigroup*, meaning that $ef = f$ for all $e, f \in C$.

(c) Show that $\alpha \in E(\mathcal{T}_X)$, the set of idempotents of $\mathcal{T}_X$ if and only if $\alpha|_{X\alpha}$ is the identity mapping on $X\alpha$.

(d) The *rank* of an element $\alpha \in \mathcal{T}_X$ is $|X\alpha|$. Let Y be a cardinal number. Show that I is an ideal of $\mathcal{T}_X$ where

$$I = \{\alpha \in \mathcal{T}_X : |X\alpha| \leq Y\}.$$

5. For a non-empty subset $A \subseteq S$ the *subsemigroup of* S *generated by* A, denoted by $\langle A \rangle$, is the smallest subsemigroup of S that contains A.

(a) Show that $\langle A \rangle$ exists and comprises the set of all products of members of A of finite length.

(b) Let $A = \{a\}$. If $\langle a \rangle$ is finite show that there exists positive integers r and m such that $\langle a \rangle = \{a, a^2, \cdots, a^{r+m-1}\}$ with $K_a = \{a^r, a^{r+1}, \cdots, a^{r+m-1}\}$ an abelian group of order m. We call r and m the *index* and the *period* respectively of the *monogenic semigroup* $\langle a \rangle$.

(c) Express the (unique) idempotent power a^t of $\langle a \rangle$ in terms of r and m.

(d) Hence show that K_a is indeed a cyclic group generated by a^{t+1}.

6. Let $X = \{0, 1, \cdots, r+m-1\}$ and consider the monogenic semigroup $S_{r,m} = \langle a \rangle$ of $\mathcal{T}_X$ where a is the map $a = (1, 2, \cdots, r+m-1, r)$ (meaning that $0a = 1$, $1a = 2$, $2a = 3, \cdots, (r+m-1)a = r$).

(a) Show that $\langle a \rangle$ has index r and period m.

(b) Find all monogenic semigroups of order 11 generated by a symbol a such that a^8 is an idempotent.

(c) Determine the subsemigroup of $\mathcal{T}_{12}$ generated by the mapping $a = (3, 3, 4, 5, 6, 7, 8, 6, 10, 11, 12, 12)$ by finding its order and period. What are its idempotents and subgroups?

7. Let S be the set of non-zero complex numbers with product $a \circ b = |a|b$.

(a) Show that S is a semigroup.

(b) Find the idempotents of S.

(c) Show that S is right simple and left cancellative.

8(a) Prove that a finite subsemigroup U of a group is a group.

(b) Show that the previous result does not hold if we delete the word 'finite'.

9. Let X be a countably infinite set and let S be the set of one-to-one mappings $\alpha : X \to X$ with the property that $|X \setminus X\alpha| = \infty$.

(a) Show that S is a subsemigroup of $\mathcal{T}_X$ (known as the *Baer-Levi semigroup* on X).

(b) Show that S is idempotent-free.

(c) Hence prove that S is right simple and right cancellative, but is not left simple nor left cancellative.

10. (a) Let S and T be two semigroups. Show that $S \times T$ is a semigroup if we define the product in the obvious way: $(s_1, t_1)(s_2, t_2) = (s_1 s_2, t_1 t_2)$.

(b) Let L and R be a *left zero semigroup* and R a right zero semigroup (see Question 4(b)). Show that the semigroup $L \times R$ consists entirely of idempotents (such a semigroup is called a *band*) in which every pair of elements comprises a mutually inverse pair.

Problem Set 2 Homomorphisms and congruences

A *semigroup homomorphism* $\alpha : S \to T$ is a mapping for which $(ab)\alpha = a\alpha b\alpha$. The definition of *monomorphism* and *isomorphism* are also defined just as for groups. We write $S \cong T$ if S and T are isomorphic.

1(a) Let $\alpha : S \to T$ be a surjective semigroup homomorphism. Let $\mathcal{A}$ denote the set of all subsemigroups of S and let $\mathcal{B}$ denote the set of all subsemigroups of T. The mapping $\phi : A \mapsto A\phi$ $(A \in \mathcal{A})$ is an inclusion-preserving map from $\mathcal{A}$ onto $\mathcal{B}$.

(b) Repeat (a), but let $\mathcal{A}$ and $\mathcal{B}$ represent the set of all ideals of S and T respectively. Draw the corresponding conclusion.

(c) Show that composition of two homomorphisms $\alpha : S \to T$ and $\beta : T \to V$, $\alpha\beta : S \to V$, is also a homomorphism.

An equivalence relation σ on a semigroup S is a *left congruence* on a semigroup S if $a\sigma b$ implies that $ca\sigma cb$ for all $c \in S$. The concept of *right congruence* is defined dually. We say that σ is a *congruence* on S of $a\sigma b$ and $c\sigma d$ then $ac\sigma bd$ for all $a, b, c, d \in S$.

2. Prove that σ is a congruence on S if and only if σ is both a left and a right congruence on S.

For any function $\alpha : S \to T$ define the *kernel of* α as $\ker(\alpha) = \{(x, y) \in S \times S : x\alpha = y\alpha\}$ and let $\text{Ker}(\alpha)$ denote the corresponding partition of S into equivalence classes. For a congruence ρ on S, we denote the set of ρ-classes of S by S/ρ; the ρ-class of $a \in S$ is written as $a\rho$.

3(a) Prove that the kernel of a homomorphism $\phi : S \to T$ is a congruence of S.

(b) Show that S/ρ is a semigroup if we define multiplication by representatives of classes, in that $a\rho b\rho = (ab)\rho$.

(c) Show that if ρ is a congruence then the *natural map* $\rho^\natural : S \to S/\rho$ for which $a \mapsto a\rho$ is a homomorphism and $\ker(\rho^\natural) = \rho$.

4. *First Isomorphism Theorem* Let $\alpha : S \to T$ be a surjective homomorphism of semigroups. Then $\rho = \ker(\alpha)$ is a congruence and there exists a unique isomorphism $\psi : S/\rho \to T$ such that $\rho^\natural\psi = \alpha$. Conversely, if ρ is any congruence on S then $\rho^\natural : S \to T$ is a surjective homomorphism of semigroups with kernel ρ.

5. *Second Isomorphism Theorem* Let σ, ρ be congruences on a semigroup S such that $\sigma \subseteq \rho$. Then

$$\rho/\sigma = \{(x\sigma, y\sigma) \in S/\sigma \times S/\sigma : (x, y) \in \rho\}$$

is a congruence on S/σ and $(S/\sigma)/(\rho/\sigma) \cong S/\rho$.

6. Let G be a group.

(a) Prove that if ρ is a congruence on G with identity e, then $N = e\rho$ is a normal subgroup of G and $a\rho b$ if and only if $ab^{-1} \in N$.

(b) Conversely show that if N is a normal subgroup of G then the relation $(a,b) \in \rho$ if and only if $ab^{-1} \in N$ is a congruence on G such that $e\rho = N$.

7(a) Show that any intersection of congruences on a semigroup S is a congruence.

(b) Hence show that given any relation $R \subseteq S \times S$, there is a smallest congruence R^* on S that contains R. We call R^* the *congruence generated by* R.

For any relation $R \subseteq S \times S$, let $R^S = R \cup R^{-1} \cup \iota$ where ι is the equality relation on S. Let $a, b \in S$ and suppose that $a = xcy$, $b = xdy$, and $cR^S d$ for some $c, d \in S$ and $x, y \in S^1$. The passage from a to b, in either direction, is an *elementary R-transition.*

8. Prove that aR^*b $(a, b \in S)$ if and only b can be obtained from a by some finite sequence of elementary R-transitions.

9. Let E be an equivalence relation on a semigroup S. Prove that the following relation is the largest congruence on S that is contained in E;

$$E^\flat = \{(a,b) \in S \times S : (\forall\, x, y \in S^1)\, (xay, xby) \in E\}.$$

10. An element $e \in S$ is a *right identity* (resp. *right zero*) if $ae = a$ (resp. $ae = e$) for all $a \in S$.

(a) Show that if S has a right identity e and a left identity f then $e = f$ is the unique identity of S.

(b) Repeat part (a) to prove the corresponding result for right and left zero elements.

(c) On any non-empty set S with may define a *null semigroup* also know as a *zero semigroup* by choosing $e \in S$ and putting $ab = e$ for all $a, b \in S$. Show that any equivalence ρ on S is a congruence and that S/ρ is also a null semigroup.

Problem Set 3 Regularity and idempotents

1. The *natural partial order* on $E(S)$, the set of all idempotents of a semigroup S: define $e \leq f$ iff $ef = fe = e$ $(e, f \in E(S))$.
(a) Verify that $\leq$ defines a partial order on $E(S)$.
(b) Show that $e \leq f$ if and only if $e = efe$.

A (lower) *semilattice* S is a poset (a partially ordered set) in which each pair of elements $a, b \in S$ has a greatest lower bound $c = a \wedge b$.

2. Show that any semilattice $(S, \wedge)$ is a commutative band (of idempotents) with respect to the meet operation and that natural partial order on S equals the partial order of the semilattice.

3. Show the converse to the result of Question 2 by proving that any commutative band B is a semilattice in which $ab = a \wedge b$, where the meet is respect to the natural partial order on B. We thus may identify the classes of semilattices and commutative bands.

4. A member $a \in S$ is called *regular* if a has an *inverse* $x \in S$ meaning that $a = axa$ and $x = xax$. The set of inverses of a is denoted by $V(a)$. A semigroup is called *regular* if all of its members are regular.
(a) Show that every group G is a regular semigroup.
(b) Show that if $axa = a$ then xax $(a, x \in S)$ is an inverse of a, and so a is regular.
(c) Show that $\mathcal{T}_X$ is a regular semigroup.
(d) Show that a homomorphic image of a regular semigroup is regular.
(e) Show that an arbitrary direct product $S = \Pi_{i \in I} S_i$ of regular semigroups is regular.

5(a) A semigroup S is a group if and only if S is regular and has a unique idempotent.
(b) A finite semigroup S is a group if and only if S is cancellative.
(c) Give an example of a semigroup that is cancellative but is not a group.

6. Any cancellative commutative semigroup S can be embedded in an abelian group as follows. Let $F = S^1 \times S^1$ and define ρ on F by $(a, b)\rho(c, d)$ if and only if $ad = bc$ $(a, b, c, d \in S^1)$.
(a) Show that ρ is congruence and that F/ρ is an abelian group.
(b) Show that S^1 is embedded in F/ρ by the mapping whereby $a \mapsto (a, 1)\rho$ $(a \in S^1)$.
(c) Carry out this process on the positive integers under addition, and on the positive integers under multiplication.

7(a) Show that if G is a group and E is a *right zero semigroup* (meaning that $ef = f$ for all $e, f \in E$) that $G \times E$ is a *right group*, which is a right simple and left cancellative semigroup.

We establish the converse of (a), which is a structure theorem for right groups, as follows.

(b) Show that $E = E(S) \neq \varnothing$;

(c) $E(S)$ is a right zero semigroup;

(d) Show that $eb = b$ for every $b \in S$ and $e \in E(S)$;

(e) Se is a subgroup of S for every idempotent e;

(f) let $f \in E(S)$ be fixed and let G be the group Sf. Prove that $S \cong G \times E$.

8. Deduce that S being a right group is equivalent to each of the following conditions:

(a) S is right simple and contains at least one idempotent;

(b) the equation $ax = b$ has a unique solution in S $(a, b \in S)$;

(c) S is regular and left cancellative.

9. Let I be an ideal of S and define ρ by $a\rho b$ if and only if $a = b$ or $a, b \in I$. Show that ρ is a congruence on S. Such a congruence is called a *Rees congruence* on S and is denoted by S/I. The class I is then the zero element of S/I.

10(a) *Cayley theorem for semigroups* Let S be a semigroup and define a mapping $\Phi : S \to \mathcal{T}_S$ by $a\Phi = \rho_a$ where ρ_a is the *right translation* by a defined by $x\rho_a = xa$. Show that Φ is a homomorphism of S into $\mathcal{T}_S$.

(b) By taking S to be a null semigroup, show that Φ is not necessarily a monomorphism.

(c) Show that by replacing S by S^1 so that $\Phi : S \to \mathcal{T}_{S^1}$, Φ becomes one-to-one and so $S\Phi$ is an isomorphic copy of S in $\mathcal{T}_{S^1}$.

Problem Set 4 Inverse semigroups

The *partial transformation semigroup* $S = \mathcal{PT}_X$. The members of S are the functions $\alpha : \text{dom}\alpha \to \text{ran}\alpha$, where $\text{dom}\alpha, \text{ran}\alpha \subseteq X$. The semigroup operation is *relational composition*, which in this instance is function composition carried out to the extent possible.

1(a) Show that for $\alpha, \beta \in \mathcal{PT}_X$ we have $\text{dom}\alpha\beta = (\text{ran}\alpha \cap \text{dom}\beta)\alpha^{-1}$ and $\text{ran}\alpha\beta = (\text{ran}\alpha \cap \text{dom}\alpha)\beta$.

(b) Let 0 be a new symbol not in X and consider $\mathcal{T}_{X\cup\{0\}}$. Prove that $\mathcal{PT}_X$ is isomorphic to the subsemigroup of all mappings in $\mathcal{T}_{X\cup\{0\}}$ that fix the point 0.

(c) If $|X| = n$, show that $|\mathcal{T}_X| = n^n|$ and $|\mathcal{PT}_X| = (n+1)^n$.

(d) $\mathcal{PT}_X$ is regular.

A regular semigroup S is an *inverse semigroup* if S is regular and the inverse of every member of S is unique. We then denote the inverse of a by a^{-1}.

2. Let $\mathcal{I}_X$ denote the subset of all one-to-one members of $\mathcal{PT}_X$. Show that $\mathcal{I}_X$ is an inverse monoid and identify its lattice of idempotents.

3. Prove that the following are equivalent for a regular semigroup S:

(i) $E(S)$ is a semilattice;

(ii) every principal right ideal and every principal left ideal has a unique idempotent generator;

(iii) S is an inverse semigroup.

4. Let S be an inverse semigroup. Show that the usual laws of inverses hold in that for $a, b \in S$ we have:

(a) $a = (a^{-1})^{-1}$ and $(ab)^{-1} = b^{-1}a^{-1}$.

(b) Also, for $e, f \in E(S)$, $Se \cap Sf = Sef$, and $Sa = Sa^{-1}a$, $Sa^{-1} = Saa^{-1}$.

The Cayley-style theorem for inverse semigroups is the *Preston-Wagner theorem*, which states that any inverse semigroup S may be embedded in $\mathcal{I}_S$ and is proved as follows.

5. Define $\Phi : S \to \mathcal{I}_S$ by $a\Phi = \rho_a$ $(a \in S)$ where we define $\rho_a : Sa^{-1} \to Sa^{-1}a$ by the rule that $x \mapsto xa$ $(x \in Sa^{-1})$. Verify that that ρ_a and $\rho_{a^{-1}}$ are mutually inverse mappings of Saa^{-1} and $Sa^{-1}a$ onto each other and conclude that $\rho_a \in \mathcal{I}_S$ and $\rho_{a^{-1}} = \rho_a^{-1}$.

6. Prove that if $\rho_a = \rho_b$ then $a = b$, so that Φ is one-to-one.

7. Show that Φ is a homomorphism and hence conclude the Preston-Wagner theorem.

8. *Lallement's lemma* Let S be a regular semigroup and ρ a congruence on S. If $a \in E(S/\rho)$ then $a\rho e$ for some $e \in E(S)$. Prove this by taking $e = axa$ where $x \in V(a^2)$.

9. Use Lallement's lemma to prove that the homomorphic image of an inverse semigroup is an inverse semigroup.

10. *Orthodox semigroups* A regular semigroup S is *orthodox* if $E(S)$ is a subsemigroup of S. In particular all bands and all inverse semigroups are orthodox. Prove that for a regular semigroup S the following are equivalent:

(i) S is orthodox;

(ii) if $a, b \in S$, $a' \in V(a)$, $b' \in V(b)$ then $b'a' \in V(ab)$;

(iii) every inverse x of an idempotent e is itself idempotent.

Furthermore, in any orthodox semigroup $aea', a'ea \in E(S)$ whenever $a' \in V(a)$ and $e \in E(S)$.

Problem Set 5 Green's relations I

Green's relations are five equivalances on a semigroup based on the notion of mutual divisibility of elements. They play no role in group theory since there they all coincide with the universal equivalence but they are important tools in the description and decomposition of semigroups.

Let S be any semigroup. We define $a\mathcal{R}b$ if $aS^1 = bS^1$ and $a\mathcal{L}b$ if $S^1a = S^1b$ $(a, b \in S)$. The equivalence $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ while the equivalence $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$, where the join is in the lattice of all equivalences of S; that is $\mathcal{D}$ is the least equivalence containing both $\mathcal{L}$ and $\mathcal{R}$. Finally, $a\mathcal{J}b$ if $S^1aS^1 = S^1bS^1$. Note that $a\mathcal{R}b$ if and only if there exist $x, y \in S^1$ such that $ax = b$ and $by = a$ with similar remarks applying to $\mathcal{L}$ and $\mathcal{J}$. The $\mathcal{L}$-class of $a \in S$ will be denoted by L_a, and similarly we have R_a, H_a, D_a and J_a for the four other Green's relations. We write $L_a \leq L_b$ if $S^1a \subseteq S^1b$ and similarly $R_a \leq R_b$ if $aS^1 \subseteq bS^1$ and $S^1aS^1 \subseteq S^1bS^1$ can be denoted by $J_a \leq J_b$.

1. Show that $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence on S.

2. Prove that every left congruence $\rho \subseteq \mathcal{R}$ commutes with every right congruence $\lambda \subseteq \mathcal{L}$, which is to say that $\rho \circ \lambda = \lambda \circ \rho$, where $\circ$ denotes relational composition.

3(a) $\mathcal{D} \subseteq \mathcal{J}$.
(b) $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$.
(c) Conclude that $a\mathcal{D}b$ if and only if there exists $c, d \in S$ such that $a\mathcal{L}c\mathcal{R}b$ and $a\mathcal{R}d\mathcal{L}b$.

Comment It follows that any $\mathcal{D}$-class D of S can be represented by an 'egg-box' diagram: a rectangular array of squares in which the rows represent $\mathcal{R}$-classes, the columns $\mathcal{L}$-classes, and the square forming the intersection of a row and a column an $\mathcal{H}$-class. Indeed we shall show that all $\mathcal{H}$-classes within the one $\mathcal{D}$-class have the same (non-zero) cardinality.

4. The set product LR of an $\mathcal{L}$-class L and an $\mathcal{R}$-class R is contained in a single $\mathcal{D}$-class.

5. Show that Green's relations on $\mathcal{T}_X$ are as follows:
(i) $\alpha\mathcal{L}\beta$ if and only if $X\alpha = X\beta$;
(ii) $\alpha\mathcal{R}\beta$ if and only if $\ker\alpha = \ker\beta$;
(iii)$\alpha\mathcal{H}\beta$ if and only if $X\alpha = X\beta$ and $\ker\alpha = \ker\beta$;
(iv) $\alpha\mathcal{D}\beta$ if and only if $\mathrm{rank}\alpha = \mathrm{rank}\beta$;
(v) $\mathcal{D} = \mathcal{J}$.

6. *Green's Lemma* (right hand version) Let $a\mathcal{R}b$ $(a, b \in S)$ and take $s, s' \in S^1$ such that $as = b$ and $bs' = a$. Then the mappings $\rho_s|L_a$ and $\rho_{s'}|L_b$ are mutually inverse, $\mathcal{R}$-class preserving bijections of L_a onto L_b and of L_b onto L_a respectively.

7. State the dual (left hand version) of Green's Lemma and hence deduce that any two $\mathcal{H}$-classes within the same $\mathcal{D}$-class are equicardinal.

8. *Miller and Clifford location theorem* For any two elements $a, b \in S$, $ab \in R_a \cap L_b$ if and only $R_a \cap L_b$ contains an idempotent. Prove this as follows.

(a) Use Green's Lemma to show that if $ab \in R_a \cap L_b$ then there exists $c \in R_b \cap L_a$ such that $cb = b$ and that $c = c^2$.

(b) Conversely take $e \in E(S) \cap R_b \cap L_a$ and show that $eb = b$ and $ae = a$.

(c) Hence use the fact that $\mathcal{R}$ and $\mathcal{L}$ are left and right congruences respectively to prove that $a\mathcal{R}ab\mathcal{L}b$.

9. Use Miller and Clifford to prove that the following are equivalent for an $\mathcal{H}$-class H of S.

(i) H contain an idempotent;

(ii) there exist $a, b \in H$ such that $ab \in H$;

(iii) H is a maximal subgroup of S.

10. Prove that any two group $\mathcal{H}$-classes H_e, H_f $(e, f \in E)$ within the same $\mathcal{D}$-class of a semigroup S are isomorphic.

Problem Set 6 Green's relations II

1. *Regular $\mathcal{D}$-classes* If one element a of a $\mathcal{D}$-class D of a semigroup S is regular then all members of D are regular, in which case D is called a *regular* $\mathcal{D}$-class. Prove this as follows.

(a) Show that if an $\mathcal{R}$- or an $\mathcal{L}$-class contains a regular element, then that class contains an idempotent.

(b) Hence prove the theorem stated above. [Hint: first prove the claim is true for R_a and for L_a.]

(c) Show that for any $a \in S$, the set of inverses $V(a) \subseteq D_a$.

2. *Inverse Location of inverses theorem* The $\mathcal{H}$-class H contains an inverse x of a if and only if $R_a \cap L_b$ and $R_b \cap L_a$ each contain an idempotent. In this case, x is the only inverse of a in H_b.

3(a) Prove that if L is a left ideal and R is a right ideal of S then $RL \subseteq R \cap L$, with equality if S is regular.

(b) If S is a right cancellative semigroup without idempotents, then every $\mathcal{L}$-class of S is trivial.

4. Let Y be a subset of X and Π a partition of X such that $|Y| = |X/\Pi|$. Let H be the $\mathcal{H}$-class of $\mathcal{T}_X$ determined by (Π, Y), meaning that $\alpha \in H$ iff $\ker\alpha = \Pi$ and $X\alpha = Y$. Then H is a group if and only if Y is a transversal of Π, in which case $H \cong \mathcal{G}_Y$, the symmetric group on Y.

Partial order of Green's classes We define $\leq_{\mathcal{L}} = \leq$ on the $\mathcal{L}$-classes of S by $L_a \leq L_b$ if $S^1a \subseteq S^1b$; similarly we define $\leq_{\mathcal{R}}$, $\leq_{\mathcal{J}}$ and $\leq_{\mathcal{H}} = \leq_{\mathcal{L}} \cap \leq_{\mathcal{R}}$. Let $\text{Reg}(S)$ denote the set of regular elements of S. We also write $a \leq_{\mathcal{L}} b$ if $L_a \leq L_b$, with a corresponding notation for the $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{J}$ partial orders on S.

5. *Hall's lemma* Let $a, b \in \text{Reg}(S)$ with $L_a \geq L_b$. Then for each $e \in E(L_a)$ there exists $f \in E(L_b)$ such that $e \geq f$ in the natural partial order. [Hint: put $f = eb'b$ where $b' \in V(b)$.]

6(a) Let U be a subsemigroup of S. Denote the Green's partial orders in U and S by $\leq_{\mathcal{L}'}$ and $\leq_{\mathcal{L}}$ etc. Let $a, b \in U$ with $b \in \text{Reg}(U)$. Then $R_a \leq_{\mathcal{R}} R_b$ implies that $R_a \leq_{\mathcal{R}'} R_b$.

(b) Let $\mathcal{G}$ denote any of $\mathcal{L}, \mathcal{R}, \mathcal{H}$ and let $\mathcal{G}'$ denote Green's relation on $U \leq S$. Prove that $\mathcal{G}' \subseteq \mathcal{G} \cap (U \times U)$ with equality if U is a regular subsemigroup of S.

7. If a regular $\mathcal{D}$-class D of S forms a subsemigroup of S then D itself has only one $\mathcal{D}$-class.

8. Prove that if S is finite then $\mathcal{D} = \mathcal{J}$.

9. Let S be a semigroup that is the union of its subgroups. Prove that each $\mathcal{D}$-class D of S is a regular subsemigroup of S and the semigroup D consists of a single $\mathcal{D}$-class (of D).

Let η denote the least semilattice congruence on a semigroup S, which is evidently given by $\eta = \eta_0^*$ where $\eta_0 = \{(a, a^2),\ (ab, ba) : a, b \in S\}$.

10(a) Show that in any semigroup $\mathcal{D}^* \subseteq \mathcal{J}^* \subseteq \eta$.

(b) Let $e, f \in E(S)$ for a regular semigroup S and let $y \in V(ef)$. Show that $fye \in V(ef) \cap E(S)$.

(c) Use part (b) to prove that in a regular semigroup, $\mathcal{D}^* = \mathcal{J}^* = \eta$. [Hint: show $\eta_0 \subseteq \mathcal{D}^*$: in order to show that $ab\mathcal{D}^*ba$ first take $a = e$, $b = f$.]

Problem Set 7: Minimal ideals and completely [0-]simple semigroups

A semigroup S is *simple* if it has just one $\mathcal{J}$-class and is *bisimple* if S has only one $\mathcal{D}$-class. If S is 0-simple if S has a zero 0, $S^2 \neq \{0\}$ and the only ideals of S are $\{0\}$ and S. A 0*-minimal ideal* M of S contains no other ideals of S apart from M and $\{0\}$ with $M \neq \{0\}$.

1(a) Show that S is simple if and only if S has only one ideal (which is necessarily S itself).

(b) Show that S is simple if and only if $S = SaS$ for all $a \in S$.

(c) Show the condition $S^2 \neq \{0\}$ serves only to exclude the two-element null semigroup from the class of 0-simple semigroups.

2. The semigroup S of part (b) below has a single $\mathcal{J}$-class but $\mathcal{D}$ is the equality relation.

(a) Show that if a semigroup S is cancellative without identity there is no pair of elements $e, a \in S$ such that $ea = a$ or $ae = a$. Deduce that in S, the $\mathcal{D}$-relation is trivial (i.e, equals the identity relation).

(b) Show that with respect to matrix multiplication:

$$S = \{\begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix} : a, b \in \mathbb{R}^+\},$$

is cancellative without identity.

(c) Show that $\mathcal{J}$ is the universal relation on S.

3(a) Show that a semigroup S either has no minimal ideals or possesses a unique minimal ideal K known as the *kernel* of S.

(b) If a semigroup S has a kernel K, then K is a simple semigroup.

(c) Show that any finite semigroup has a simple kernel.

(d) S is 0-simple if and only if$SaS = S$ for every $a \in S \setminus \{0\}$.

4. By a 0-minimal ideal M of S we mean that M is an ideal of S, $M \neq \{0\}$, and that M contains no ideals of S other than $\{0\}$ and itself. Prove that if M is a 0-minimal ideal of S then either $M^2 = \{0\}$ or M is a 0-simple semigroup.

5. If I, J are ideals of S such that $I \subseteq J$ and there is no ideal of S lying strictly between I and J, then J/I is either 0-simple or null.

6. Prove that if $J_a \in S/\mathcal{J}$ then either J_a is the kernel of S or the set $I = \{x \in S : J_x < J\}$ is an ideal of S (and hence of $J(a) = S^1aS^1$) and hence the *factor* $J(a)/I$ is either 0-simple or null.

Comment: The semigroups K and $J(a)/I(a)$ are called the *principal factors* of S. A semigroup is called *semisimple* in none of its principal factors are null. A principal factor J/I can be thought of as the $\mathcal{J}$-class J together with 0 and for any $a, b \in J$, the product of a and b is ab if $ab \in J$ and is 0 otherwise.

A 0-simple semigroup S is called *completely* 0*-simple* if S contains a *primitive idempotent* e, which means $e \neq 0$ and $f \leq e$ $(e, f \in E(S))$ then $f \in \{0, e\}$.

A simple semigroup is *completely simple* if S^0 is completely 0-simple. A semigroup S is *completely regular* if every element a has an inverse x with which it commutes.

7(a) Prove that S is completely regular if and only if S is a union of its subgroups.

(b) Prove that each $\mathcal{D}$-class of a completely regular semigroup is a completely simple semigroup and a union of isomorphic groups.

8. Let S be a completely regular semigroup.

(a) Show that $\mathcal{J}$ is a congruence on S and that $S/\mathcal{J}$ is a semilattice;

(b) Hence deduce that $\mathcal{J} = \eta$, the least semilattice congruence on S.

9. Show that any simple completely regular semigroup is completely simple.

A semigroup S is called a *semilattice of semigroups of type* T if there is a congruence ρ on S such that S/ρ is a semilattice and each class $a\rho$ is a subsemigroup of S of type T. (Since S/ρ is a band, it follows that all ρ classes $a\rho$ are subsemigroups of S as $(a\rho)^2 = a^2\rho = a\rho$.)

10. Prove that $\mathcal{D} = \mathcal{J}$ in any completely regular semigroup S and that S is a semilattice of completely simple semigroups.

Problem Set 8

Let Y be a semilattice and let $\{S_\alpha : \alpha \in Y\}$ be a collection of disjoint semigroups of the same type T, indexed by Y. Suppose that for each $\alpha, \beta \in Y$ such that $\alpha \geq \beta$ there is a homomorphism $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ such that:

(i) $\phi_{\alpha,\alpha}$ is the identity mapping of S_α;

(ii) $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ for every $\alpha, \beta, \gamma \in Y$ with $\alpha \geq \beta \geq \gamma$. We then define an associative product on $S = \{S_\alpha : \alpha \in Y\}$ by

$$a_\alpha \circ b_\beta = (a_\alpha \phi_{\alpha,\alpha\beta})(b_\beta \phi_{\beta,\alpha\beta});\ a_\alpha \in S_\alpha,\ b_\beta \in S_\beta.$$

1. Show that the following are equivalent:

(i) S is regular and every idempotent is *central* (meaning that $ea = ae$ for all $e \in E(S)$, $a \in S$);

(ii) every $\mathcal{D}$-class of S has a unique idempotent;

(iii) S is a semilattice of groups;

(iv) S is a *strong semilattice of groups.*

2. Show that a semigroup S is a semilattice of groups if and only if

$$(\forall a, b \in S)(\exists x, y \in S) :\ (a = axa) \wedge (ab = bya).$$

3. Prove that a commutative semigroup S is regular if and only if S is a strong semilattice of abelian groups.

4. A semigroup S is called a *rectangular band* if it satisfies $a = aba$ $(\forall a, b \in S)$. Show this condition is equivalent to be *nowhere commutative*, meaning that $ab = ba$ implies $a = b$.

5(a) Let L, R be non-empty sets and define a product on $L \times R$ by $(a, b)(c, d) = (a, d)$. Verify that this product is that of a rectangular band.

(b) Conversely show that any rectangular band is isomorphic to a rectangular band of the type described in (a).

6. Prove that any band is a semilattice of rectangular bands.

7. We say that $\langle X, R\rangle$ is a *presentation* for semigroup S if X is a generating set of S and $S = F_X/\rho_R$ where ρ_R is the congruence on the free semigroup F_X generated by the set of pairs $R \subseteq X \times X$. We write $x = 1$ (resp. $x = 0$) to denote the fact that $xa = ax = a$ (resp. $xa = ax = x$) $\forall\, a \in S$.

Show that the semigroup defined by the presentation $\langle x, y | xyx = 1\rangle$ is a group isomorphic to the integers under addition.

8. *The bicyclic monoid* Let $M = \langle a, b | ab = 1\rangle$.

(a) Let $S = \langle \alpha, \beta\rangle$, where $\alpha, \beta \in \mathcal{T}_{\mathbb{N}^0}$ are the mappings defined by $n\alpha = n + 1$, $n\beta = \max\{n - 1, 0\}$ for all $n \in \mathbb{N}^0$. Show that $\alpha\beta = 1$ but that $\beta\alpha \neq 1$. Deduce that S is a homomorphic image of M.

(b) Show that any member of S, and hence of M, can be uniquely expressed in the form $b^m a^n$. Deduce that S is a faithful representation of M.

9(a) For M as in Question 8, show that

$$b^k a^l \cdot b^m a^n = b^i a^j, \text{ where } i = k + m - \min\{l, m\}, \ j = l + n - \min(l, m).$$

(b) $b^m a^n \in E(M)$ if and only if $m = n$.

10 (a) Show that the $\mathcal{R}$-and $\mathcal{L}$-classes of M are respectively the sets of the form $R_{b^i} = \{b^i a^j : 0 \leq j\}\ i \geq 0\}$, $L_{a^j} = \{b^i a^j : 0 \leq i\}\ j \geq 0$ and that $\mathcal{H}$-classes are all singletons. Conclude that M is a bisimple monoid.

(b) Show that M is an inverse semigroup and that the semilattice of idempotents of M is an infinite descending chain.

Problem Set 9 Completely 0-simple semigroups

Let S be a semigroup with zero 0. Then $e \in E(S)$ is *primitive* if e is 0-*minimal* meaning that if $f \in E(S)$ with $f \leq e$ then $f = 0$ or $f = e$. A semigroup S is *completely* 0*-simple* if S is 0-simple with a primitive idempotent. A semigroup is *completely simple* if it simple with a primitive idempotent.

1. Let S be a finite 0-simple semigroup.
(a) Show that S is completely simple.
(a) Show that S is regular.

2. Continue with the finite 0-simple semigroup of Question 1, with non-zero $\mathcal{D}$-class D.
(a) Use the extended right regular representation of S in $\mathcal{T}_{S^1}$ to prove that if $a, b \in D$ then either $ab = 0$ or $a\mathcal{R}ab\mathcal{L}b$ in S.
(b) Deduce that if $ab \neq 0$ then $L_a \cap R_b$ is a group.

Index the rows and columns of D by I and Λ respectively and without loss of generality assume that $(1, 1) \in I \times \Lambda$ with $H_{1,1}$ a group. For each $i \in I$ and $\lambda \in \Lambda$ choose a fixed $r_i \in H_{i,1}$ and $q_\lambda \in H_{1,\lambda}$.
3(a) Prove that the mapping whereby $a \mapsto r_i a q_\lambda$ defines a bijection $\phi_{i,\lambda} : H_{1,1} \mapsto H_{i,\lambda}$.
(b) Deduce from part (a) that each $x \in H_{i,\lambda}$ can be represented by a triple $(a; i, \lambda)$ where $a \in H_{1,1}$ and $x = r_i a q_\lambda$.
(c) Let $y \in H_{j,\mu}$ say, with representation $(b; j, \mu)$. By identifying each of x and y with their representation triple as in part (b), show that, for some $c \in H_{1,1}$,

$$xy = (a; i, \lambda)(b; j, \mu) = (c; i, \lambda).$$

Note If $xy = 0$, which occurs if and only if $H_{j,\mu}$ is not a group, we take $c = 0$ and agree that $(0; i, \lambda)$ represents the zero of S for all $(i, \lambda) \in I \times \Lambda$.

Rees matrix semigroups Let I, Λ be index sets and let G^0 be a group with adjoined zero 0. (We call G^0 a *group with zero*). Let $P = (p_{\lambda,i})$ be a $\Lambda \times I$ matrix with entries from G^0. The *Rees matrix semigroup* with *sandwich matrix* P is the set $S = M^0[G; I, \Lambda, P]$ where:

$$S = \{(a; i, \lambda) : a \in G^0, i \in I, \lambda \in \Lambda\} \cup \{0\},$$

with product

$$(a; i, \lambda)(b; j, \mu) = (ap_{\lambda j}b; i, \mu)$$

with the understanding that $(0; i, \lambda) = 0$ for all $i \in I, \lambda \in \Lambda$ and that any product involving 0 is itself 0.

4(a) Prove that a Rees matrix semigroup S is indeed a semigroup.
(b) Show that S is regular if and only if P is *regular* in the sense that every row and column of P contains a non-zero entry.

(c) Conclude that every finite 0-simple semigroup is isomorphic to a regular Rees matrix semigroup $M^0[H_{1,1}; I, \Lambda, P]$.

5. Let $R = R_e$ denote the $\mathcal{R}$-class of a primitive idempotent of an arbitrary completely 0-simple semigroup S.

(a) Let $b \in eS$ and write $e = xby$ (why is that justified?). Show that $f = byexe \in E(S)$ with $f \leq e$.

(b) Show that $xfby = e$ and hence deduce that $f = e$, using the fact that e is a primitive idempotent.

(c) Hence prove that $R \cup \{0\}$ is the right ideal eS of S.

(d) Furthermore, deduce that $R \cup \{0\}$ is a 0-minimal right ideal of S.

6. Continue under the hypotheses of Question 5.

(a) Prove that for any $x \in S$, $R_x \cup \{0\} = c(R \cup \{0\}) = ceS$ for some $c \in S$.

(b) Prove that $R_x \cup \{0\}$ is a minimal right ideal for every $x \in S$

7(a) Show that any completely 0-simple semigroup S is regular and 0-bisimple (with a non-zero $\mathcal{D}$-class D).

(b) Moreover, for $a, b \in D$, either $a\mathcal{R}ab\mathcal{L}b$ or $ab = 0$.

(c) Conclude that a semigroup S is completely 0-simple if and only if S is ismorphic to some regular Rees matrix semigroup.

8. It is always possible to represent a completely 0-simple semigroup $S = M^0[G; I, \Lambda, P]$ so that a given row λ and column i of P consists entirely of 0 and e, the identity element of the group G.

9. Verify that any completely 0-simple semigroup S has the following properties:

(i) every non-zero idempotent is primitive;

(ii) $\mathcal{H}$ is a congruence on S;

(iii) any non-trivial homomorphic image of S is completely 0-simple.

10. A completely 0-simple inverse semigroup is called a *Brandt semigroup.* A semigroup is Brandt if and only if S is isomorphic to a Rees semigroup of the form $M^0[G; I, I, \Delta]$ where Δ is the $I \times I$ identity matrix.

Problem Set 10 Properties of completely 0-simple semigroups

1. Prove that a regular semigroup S has all non-zero idempotents primitive if and only if S is a 0-direct union of completely 0-simple semigroups.

[Hint: Suppose that $\{0\} \neq J_f \leq J_e$ $(e, f \in E)$. Then $f = xey$ say; put $g = eyfxe$ and show that $g = e$.]

2. Show that the following are equivalent for a regular semigroup S:
(i) S is completely simple;
(ii) $aba = a$ implies $bab = b$ for all $a, b \in S$;
(iii) S is *weakly cancellative*, meaning that $ax = bx$ and $ya = yb$ for some $x, y \in S$ implies that $a = b$.

3. A completely simple semigroup S is orthodox if and only if S is a *rectangular group*, meaning a direct product of a group and a rectangular band.

4. A completely 0-semigroup with trivial maximal subgroups is called a *0-rectangular band.* Show that S is a 0-rectangular band if and only if S has a zero 0 and satisfies the two conditions:
(i) $xyx = x$ or $xyx = 0$ for all $x, y \in S$;
(ii) $xSy = \{0\}$ implies $x = 0$ or $y = 0$.

Miscellaneous exercises

5. Show that each of the following binary operations are associative on $\mathbb{R}^{>0}$, the set of positive real numbers.
(i) $x \circ y = \sqrt{x^2 + y^2}$.
(ii) $x \circ y = \frac{xy}{x+y}$.
(iii) $x \circ y = \ln(e^x + e^y - 2)$.

6. Let S be a semigroup and $f : S \to S$ any permutation of S. We denote the binary operation of S by $+$ without assuming that S is commutative. Define a binary operation $\circ$ on S by

$$x \circ y = f^{-1}(f(x) + f(y)).$$

(a) Prove that $(S, \circ)$ is a semigroup.
(b) Show that $f : (S, \circ) \to (S, +)$ is an isomorphism.
(c) Show that each of the three operations of Question 5 is of this type.

The *free semigroup* F_X on a set X is the set of all *words* or *strings* $x_1 x_2 \cdots x_n$ $(n \geq 1)$ where $x_i \in X$.

7(a) Show that F_X is free on X in the algebraic sense that if S is any semigroup and $\alpha : X \to S$ is any function then there is a unique homomorphism $\phi : F_X \to S$ such that $\iota\phi : X \to S$ is equal to α, where $\iota : X \to F_X$ is an embedding of X into F_X.
(b) Prove that any semigroup is a homomorphic image of some free semigroup.

(c) Show that F_X is unique in that if G is another semigroup with the defining property of part (a), then F_X and G are isomorphic.

8. Let $\phi : S \to T$ be a surjective homomorphism from a finite semigroup and let G be a subgroup of T. Let U be a subsemigroup of S of least cardinal such that $U\phi = G$. Prove that U is a subgroup of S.

9. Generalization of Lallement's Lemma due to T.E. Hall (Set 4 Question 8) Let $\phi : S \to T$ be a surjective homomorphism from a regular semigroup S. Suppose that $(c, d) \in V(T)$. Then there exists $(a, b) \in V(S)$ such that $a\phi = c$ and $b\phi = d$. Prove this as follows.

(a) Take $x, y \in S$ such that $x\phi = c$ and $y\phi = d$ and $(xyxy)' \in V(xyxyx)$. Put $a = xy(xyxy)'yx$, $b = y(xyxy)'xy$. Show that $b \in V(a)$.

(b) Show that $(xyxy)\phi \in V((cd)^2) = V(cd)$ and that a and b are the required elements of S that satisfy our claim.

(c) Prove that this result implies Lallement's lemma.

10. Let $\alpha, \beta : (\mathbb{Z}, +) \to T$ be two semigroup homomorphisms such that $n\alpha = n\beta$ for all $n \geq 1$. Prove that $\alpha = \beta$.

Hints for Problems

Problem Set 1

2. For the converse, first use left and right simplicity to show that S is a monoid.

5(c) Show you are after the least integer $p \geq r$ such that $m|p$.

(d) a^{t+1} is a generator of K_a.

Problem Set 2

1(a) For $U \leq T$, show that $U\alpha^{-1} \leq S$ and $(U\alpha^{-1})\alpha = U$.

Problem Set 3

3. (iii) implies (i) (iii) implies (i). Let $e, f \in E(S)$ and $x = (ef)^{-1}$ then xe and fx are both inverse to ef. Now show $x = x^2$ and then $x = ef$. Deduce $ef, fe \in E(S)$ and finally that $ef = fe$.

7(f) Show that each $a \in S$ has a unique right identity element e and then use the mapping whereby $a \mapsto (af, e)$.

10. (ii) implies (iii). Since $xe, ex \in E(S)$ it follows from the given property that $ex^2e \in V(xe^2x)$. But $x = xe^2x$, which is inverse to ex^2e, and thus

$$x = x(ex^2e)x = (xex)(xex) = x^2.$$

(iii) implies (i). Let $e, f \in E(S)$ and take $x \in V(ef)$. Show that $ef \in V(fxe)$ and $fxe \in E(S)$.

Problem Set 5

4. Use the fact that $\mathcal{L}$ and $\mathcal{R}$ are right and left congruences respectively.

10. A required isomorphism is $x \mapsto a'xa$ where $a'a = f$.

Problem Set 6

2. In the reverse direction, show and then use that ρ_a defines a bijection of L_b onto L_a.

7. Apply Question 6(b) to the subsemigroup D.

8. If $a\mathcal{J}b$ we may write a and b in the forms $a = (ux)^n a(yv)^n$ and $b = (xu)^n b(vy)^n$ and n may be chosen so that $a(yv)^n = a$; and $a = (ux)^n a$, $b = (xu)^n b = b(vy)^n$. Put $c = xa$ show that $a\mathcal{L}c\mathcal{R}b$.

Problem Set 7

9. Suppose that $e, f \in E(S)$ with $f \leq e$. Take $z, t \in S$ such that $e = zft$. Put $x = ezf$ and $y = fte$. Show $xfy = e$ and $ex = xf = x$, $fy = ye = y$. Take $x \in H_g$ $(g \in E(S))$. Tthere exists $x* \in H_g$ such that $xx* = x * x = g$. Show $g = gf = f$ and so $e = f$.

10. Make use of Questions 6 and 9.

Problem Set 8

Problem Set 9

7. Show first that $aSb \neq 0$ for all $a, b \in S$. The use Question 6 to show that $a\mathcal{R}c\mathcal{L}b$.

Answers to the Problems

Problem Set 1

6. (b) $S_{8,4}$ and $S_{4,8}$. (c) $\langle a \rangle = S_{4,3}$. The idempotent of $\langle a \rangle$ is a^6. $K_a = \{a^4, a^5, a^6 = e\} \cong \mathbb{Z}_3$.