Mathematics 102 Geometry & Trigonometry

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Solutions and Comments for the Problems

Problem Set 1

1. Let x denote the length of the shorter side. The given information is captured by the equation:

$$x^2+(3x-1)^2 = (3x+1)^2 \Rightarrow x^2 = (3x+1)^2-(3x-1)^2 = (3x+1+3x-1)(3x+1-(3x-1))$$

$$\Rightarrow x^2 = 6x(3x+1-3x+1) = (6x)(2) = 12x \Rightarrow x^2-12x = x(x-12) = 0 \Rightarrow x = 12.$$

Therefore the length of the hypotenuse is $3x + 1 = 37$ inches; the other side has length $3x - 1 = 35$ inches.

2. Let y denote the unknown length. We calculate the area of the triangle in two ways to give:

$$A = \frac{1}{2} \cdot 12 \cdot 35 = \frac{1}{2} \cdot 37y \Rightarrow y = \frac{12 \cdot 35}{37} = 11 \cdot 35\,\text{inches}.$$

3. We work in units of 10^6 for convenience. The question tells us that M is the centre of a semicircle with AB as base. Since the angle in a semicircle is a right angle, we have that $\angle ACB = 90°$ and we may apply Pythagoras's Theorem. We get $|AB|^2 = 5^2 + 12^2 = 169$. Hence $|AB| = 13 \times 10^6$,which is to say that $|AB| = 1.3 \times 10^7$km.

Comment That the angle in a semicircle is a right-angle is of course a special case of the classical Euclidean Circle Theorem that the 'angle at the centre of a circle is twice that on the circumference'. This means that if A and B are two points at either end of a chord of a circle with centre O, and C is *any* point on the circumference then $\angle AOB = 2\angle ACB$. The semicircle theorem is then just the special case that arises when that chord is the diameter of the circle.

4. $|OP|^2 = 3^2 + 4^2 = 9 + 16 = 25 \Rightarrow |OP| = 5$. Since a tangent touches the circle at right angles to the radius at that point it follows that $\triangle PQO$ has $\angle PQO = 90°$ and we may apply Pythagoras and the fact that $|OQ| = 1$ to infer that $|QP|^2 = 5^2 - 1^2 = 25 - 1 = 24$. Therefore $|QP| = \sqrt{24} = \sqrt{4 \times 6} = \sqrt{4} \times \sqrt{6} = 2\sqrt{6}$.

Comment Note that there are two (symmetric) possibilities for the line L.

5. Let OA be the radius of the circle that passes through A. Now

$$|OA|^2 = \left(\frac{1}{2}\right)^2 + \left(-\frac{2}{3}\right)^2 = \frac{1}{4} + \frac{4}{9} = \frac{9+16}{36} = \frac{25}{36} \Rightarrow |OA| = \frac{5}{6}.$$

The required length is $|AB| = 1 - \frac{5}{6} = \frac{1}{6}$.

Comment We are saying implicitly that for any point C on the circle, $|AC| > |AB|$, which is true because the lengths of two sides of any triangle exceed the length of the third side and so $|OA| + |AC| > |OC| = 1$ whence $|AC| > 1 - |OA| = |AB|$.

6. Let r be the radius of the planet. The initial length of the cable is $2\pi r$. The extended cable has length $2\pi(r+1)$, so that in both cases the increase is

$$2\pi(r+1) - 2\pi r = 2\pi \text{ metres.}$$

Comment People are often surprised that the outcome is independent of the radius of the circle. Even more surprising perhaps is that we get the same result for *any* smooth closed curve C: at each point of C imagine we erect a normal line segment of length 1 unit and let C' be the curve that is traced out be the end of the normal as ths base traverses the curve C. The curve C' is then an enlarged version of C and it may be shown that the increase in length as we pass from C to C' is always 2π. This is a classic problem in *arc lengths of curves*.

7. Opposite angles in a *concyclic quadrilateral* sum to $180°$. Hence $\angle BCD = 180° - 110° = 70°$ and $\angle CDA = 180° - 40° = 140°$.

Comment Any three corners of a quadrilateral will lie on the radius of a circle (whose centre is the intersection of the perpendicular bisectors of the corresponding sides). In general there is no reason why the fourth corner should also lie on that circle but if it does we have a concyclic quadrilateral and a simple geometric argument using the circle and isosceles triangles shows that the opposite angles are *supplementary*, that is each such pair sum to $180°$.

8. Let O be the centre of the circumscribing circle of the regular polygon. Then $\angle AOB = \frac{2\pi}{11}$ and so $\angle ACB = \frac{1}{2}\angle AOB = \frac{1}{2} \cdot \frac{2\pi}{11} = \frac{\pi}{11}$.

9. Let θ be the required angle. Partitioning the n-gon into n triangles with the centre as common vertex and summing their interior angles yields:

$$n\pi = n\theta + 2\pi \Rightarrow \theta = \Big(\frac{n-2}{n}\Big)\pi.$$

(The $n\pi$ term arises as the sum of the angles of each of the n triangles is π and the 2π term represents the sum of all the angles of those triangles at the common vertex that is the centre.)

10. The (equilateral) triangle, the square, and the hexagon. No others are possible, for if k n-gons meet at a vertex then

$$k\Big(\frac{n-2}{n}\Big) = 2\pi \Rightarrow k = \frac{2n}{n-2} \in \mathbb{Z}^+.$$

But $2 < \frac{2n}{n-2} < 3 \,\forall\, n \geq 7$; and for $n = 5$, $\frac{2n}{n-2} = \frac{10}{3} \notin \mathbb{Z}$.

Comment Indeed any triangle tessellates (by parallelograms made up of of pairs of the given triangle) and, much less obviously, so does any quadrilateral. (See *Mathematics for the Imagination*).

Problem Set 2

1. The dodecahedron has 12 pentagonal faces, with each edge meeting 2 faces and therefore:

$$\#\text{edges} = \tfrac{1}{2}(12 \times 5) = 30.$$

2. The regular solid will have 6 vertices, with each face an equilateral triangle, giving an octahedron.

3. The regular solid will have 8 vertices, with each face a square, giving a cube.

Comment: the cube and octahedron are *duals* of one another in that the operation of forming a solid by taking a vertex at the centre of each face, transforms one into the other.

4. $CD \parallel AB$ so the equation of CD is $4x + 3y = k$ for some k. Substituting the co-ordinates of the point $C = (-1, -1)$ into this equation gives:

$$4(-1) + 3(-1) = k \Rightarrow k = - - 4 - 3 = 7;$$

Therefore the required equation of the line is $4x + 3y + 7 = 0$ or $y = -\frac{4}{3}x - \frac{7}{3}$.

5. Joining the centres of the circles gives an equilateral triangle T of side length 2 and so area $\frac{1}{2} \cdot 2 \cdot \tan 60^\circ = \sqrt{3}$. Each circle meets T in a sector S of angle 60° so the required area is represented by $T - 3S$

$$= \sqrt{3} - 3 \cdot \frac{60}{360} \cdot \pi = \sqrt{3} - \frac{\pi}{2} \approx 0 \cdot 1613.$$

6. The radius of the maximum circle is given by $\tan 30^\circ$ so the required area of the circle is

$$\pi(\frac{\sqrt{3}}{3})^2 = \frac{\pi}{3}.$$

7. $\triangle AC_1C_2$ is equilateral, all sides being radii of unit circles. We first find the area of the segment AC_1B, which equals

$$(\text{Area of sector } ABC_2) - (\text{Area of } \triangle ABC_2) =$$

$$\frac{\pi}{3} - 2\text{Area}\triangle AC_2D = \tfrac{\pi}{3} - 2 \cdot \tfrac{1}{2} \sin \tfrac{\pi}{3} \cos \tfrac{\pi}{3} =$$

$$\frac{\pi}{3} - \frac{1}{2} \sin \frac{\pi}{3} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}.$$

Therefore the total required area is:

$$\frac{2\pi}{3} - \frac{\sqrt{3}}{2} = \frac{4\pi - 3\sqrt{3}}{6}.$$

8. We see that the action of the pair of mappings is described by $(x, y) \mapsto (x, -y) \mapsto (-x, -y)$, which corresponds to a rotation of $180°$ about the origin.

9. The interior angle is given by $\left(\frac{n-2}{n}\right)\pi$ (see Set 1, Question 9) and so the exterior angle is

$$\pi - \frac{n-2}{n}\pi = \pi(\frac{n-n+2}{n}) = \frac{2\pi}{n}.$$

10. It follows that the sum of all the exterior angles is $n \cdot \frac{2\pi}{n} = 2\pi$, one complete turning.

Problem Set 3

1(a) $\frac{2\pi}{4} = \frac{\pi}{2}$.
(b)

$$y = 3 + 3\sin 3x \cos 3x = 3 + \frac{3}{2}\sin 6x,$$

hence the period of y is $\frac{2\pi}{6} = \frac{\pi}{3}$ and the maximum value of y is $3 + \frac{3}{2} = \frac{9}{2}$.

2. Since $\arctan 1 = \frac{\pi}{4}$ we turn to the remaining two terms. We use the identity

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}:$$

$$\tan(\arctan 2 + \arctan 3) = \frac{2+3}{1-(2)(3)} = \frac{5}{-5} = -1;$$

taking the arctan of both sides gives $\arctan 2 + \arctan 3 = \arctan(-1) = \frac{3\pi}{4}$ and therefore:

$$\arctan 1 + \arctan 2 + \arctan 3 = \frac{\pi}{4} + \frac{3\pi}{4} = \pi.$$

Comment We need to be mindful that we may only say immediately that $\arctan 2 + \arctan 3 = \frac{3\pi}{4} + m\pi$, but we observe that $0 < \arctan 2 + \arctan 3 < \pi$, so that $m = 0$ here.

OR

$$\tan(\arctan 1 + \arctan 2) = \frac{1+2}{1-(1)(2)} = -3$$

and so

$$\tan((\arctan 1 + \arctan 2) + \arctan 3) = \frac{-3+3}{1-(-3)(3)} = 0,$$

which then allows the conclusion that the three arctans sum to π.

OR If you are familiar with complex numbers we multiply together $1+i$,$1+2i$, and $1+3i$ (as their respective arguments are $\arctan 1$, $\arctan 2$, and $\arctan 3$) and we get:

$$(1+i)(1+2i)(1+3i) = (-1+3i)(1+3i) = (3i)^2 - 1^2 = -9 - 1 = -10.$$

The argument of a product is the sum of the arguments, and since -10 is on the negative real line, its argument is π, and so we again have the result.

3. Put $\sin\theta + \cos\theta = r\cos(\theta-\alpha) = r(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$; equating coefficients gives $r\cos\alpha = r\sin\alpha = 1$, whereupon squaring gives

$$r^2\cos^2\alpha + r^2\sin^2\alpha = r^2(\cos^2\alpha+\sin^2\alpha) = r^2 = 1^2+1^2 \Rightarrow r = \sqrt{2};$$

$$\sin\alpha = \cos\alpha = \frac{1}{\sqrt{2}} \Rightarrow \alpha = \frac{\pi}{4};$$

$$\sqrt{2}\cos(\theta-\frac{\pi}{4}) = \sqrt{2} \Rightarrow \cos(\theta-\frac{\pi}{4}) = 1\ (0\le\theta\le 2\pi)$$

$$\Rightarrow \theta - \frac{\pi}{4} = 0 \Rightarrow \theta = \frac{\pi}{4}.$$

Comment The conversion of a sum of sines and cosines to a single cosine function is very useful as it allows you to see many important aspects of the function at once. For example the amplitude r gives the maximum and minimum values of the function and the lag α allows you to see where these turning points occur, without the need for any calculus. Students often solve this equation by first squaring both sides. Squaring is not a one-to-one operation so will generally drag in extraneous solutions: in this case it leads to the equation $\sin 2\theta = 1$, which in the specified range $0 \le \theta \le 2\pi$, has two solutions. These are $\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$ but only the former is a valid solution of the original equation.

4. Since $\sec^2 x = 1+\tan^2 x$ the equation can be rewritten as $\tan^2 x + 5\tan x + 3 = 0$. Solving for $\tan x$ yields:

$$\tan x = \frac{-5\pm\sqrt{13}}{2} = -0\cdot 6972,\ -4.3028.$$

Taking the inverse tan function of each of these values then yields the approximate second quadrant solutions as 2.53 and 1.80 respectively.

5.

$$\sec\left(\theta+\frac{\pi}{6}\right) = 2 \Leftrightarrow \cos\left(\theta+\frac{\pi}{6}\right) = \frac{1}{2} \Leftrightarrow \theta + \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3}$$

$$\theta = 2n\pi + \frac{\pi}{6},\ \text{or}\ 2n\pi - \tfrac{\pi}{2}\ (n\in\mathbb{Z}).$$

6. Let $\theta = \arccos\left(\frac{9}{41}\right)$. Then take the right triangle with hypotenuse of 41 and one side of length 9 so that the included angle is θ. Marking the other side as y we get $y^2 = 41^2 - 9^2 = 1600 \Rightarrow y = 40$. Hence $\sin\theta = \sin(\arccos\left(\frac{9}{41}\right)) = \frac{40}{41}$.

7. Applying the Cosine Rule yields the equation: $(x+y)^2 = x^2 + (x-y)^2 - 2x(x-y)\cos A \Rightarrow 4xy - x^2 = -2x(x+y)\cos A$

$$\Rightarrow \cos A = \frac{x-4y}{2(x-y)}.$$

8. The Sine Rule here gives: $\frac{\sin A}{x+y} = \frac{\sin B}{x} = \frac{\sin C}{x-y}$. Hence

$$\sin A - 2\sin B + \sin C = \sin B\Big(\frac{x+y}{x} - 2 + \frac{x-y}{x}\Big) = 0.$$

9. We require the angle at the vertex B and since we have all three sides of the triangle we re-arrange the Cosine Rule to make $\cos B$ the subject and then insert the given values:

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{(5\cdot 2)^2 + (7\cdot 1)^2 - (3\cdot 7)^2}{2(5\cdot 2)(7\cdot 1)} \approx 0\cdot 8635.$$

Therefore $B \approx 30.29°$, which, to the nearest degree gives, $B = 30°$.

Comment It is always easier to do the necessary algebraic manipulation first, thereby giving yourself a formula for the required value, and then to substitute the numbers at the very end of the exercise to gain the answer. Students often feel on safer ground when the calculator takes over and so insert numbers at the earliest possible opportunity but then struggle to deal with the messy number manipulations that follow.

10. Let $AB = 10$, $BC = 9$, and $\angle CAB = 60°$. We have from the Cosine rule:

$$\begin{aligned} BC^2 &= AB^2 + AC^2 - 2(AB)(AC)\cos\angle CAB \\ \Rightarrow 9^2 &= 10^2 + AC^2 - 2(10)(AC)\cos 60° \\ \Rightarrow AC^2 &- 10AC + 19 = 0 \\ \Rightarrow AC &= \frac{10 \pm \sqrt{100-76}}{2} = \frac{10 \pm 2\sqrt{6}}{2} = 5 \pm \sqrt{6}. \end{aligned}$$

Comment This case where we are given two sides and an angle that does *not* lie between these sides is called the *ambiguous case* as the given information in general leads to two distinct possibilities for the third side.

Problem Set 4

1. $\sin(x + \frac{\pi}{4}) = \sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(\sin x + \cos x)$. Now since the sine function is odd, $\int_{-t}^{t} \sin x\, dx = 0$, giving:

$$\frac{1}{\sqrt{2}} \frac{\int_{-t}^{t} \cos x\, dx}{\int_{-t}^{t} \cos x\, dx} = \frac{1}{\sqrt{2}}.$$

Comments Remember that a function $f(x)$ is *even* if $f(-x) = f(x)$ for all x, (examples are any even power of x and the cosine function) and $f(x)$ is *odd* if $f(-x) = -f(x)$ for all x (examples being all odd powers of x and the sine function). You should be able to discover for yourself simple rules concerning the sums, differences, products and quotients of even and odd functions: for example, the quotient of an even by an odd is odd, so $\tan x$ is an odd function. For any odd function we have by symmetry that $\int_{-t}^{t} f(x)\,dx = 0$ while for an even function we get $\int(_{-t}^{t} f(x)\,dx = 2\int_0^t f(x)\,dx$. Both the facts are 'obvious' from the picture of the corresponding graph although it is a good exercise to verify them algebraically. In any event, they are symmetries that often simplify definite integrals considerably, as in this case where, in the end, no integration at all was needed.

2. Since the period of the tan function is π we get here $\frac{\pi}{5}$ for the period of $y(x)$.

3. Put $A = 45°$, $B = 30°$ in the given identity to obtain:

$$\tan 75° = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{4+2\sqrt{3}}{3-1}$$

$$= 2 + \sqrt{3}.$$

4.

$$\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30° = \frac{\sqrt{2}}{2}(\frac{\sqrt{3}}{2} + \frac{1}{2}) =$$

$$= \frac{\sqrt{2}}{4}(\sqrt{3}+1) = \frac{\sqrt{6}+\sqrt{2}}{4}.$$

Or, applying the alternative formula gives

$$\cos^2 15° = \frac{1}{2}(1 + \cos 30°) \text{ and so } \cos^2 15° = \frac{1}{2}(1 + \frac{\sqrt{3}}{2}) = \frac{2+\sqrt{3}}{4}.$$

$$\text{whence } \cos 15° = \frac{1}{2}\sqrt{2+\sqrt{3}}.$$

Again the two different looking answers may be directly reconciled through comparing their squares or by use of the formula

$$\sqrt{A \pm \sqrt{B}} = \sqrt{\frac{A+C}{2}} \pm \sqrt{\frac{A-C}{2}}, \text{ where } C = \sqrt{A^2 - B}.$$

5. If we differentiate the function $y = \arcsin x + \arccos x$ we get $y\prime = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$, so that $y = c$, a constant. If we put $x = 0$ we see that $c = \arcsin 0 + \arccos 0 = 0 + \frac{\pi}{2} = \frac{\pi}{2}$.

$$\therefore \arcsin x + \arccos x = \frac{\pi}{2}.$$

Alternatively, call the quantity A and consider

$$\sin A = \sin(\arcsin x + \arccos x) = \sin(\arcsin x)\cos(\arccos x) + \cos(\arcsin x)\sin(\arccos x)$$

$$= x^2 + \left(\sqrt{1-x^2}\right)\left(\sqrt{1-x^2}\right) = x^2 + (1-x^2) = 1,$$

and so $A = \frac{\pi}{2}$.

Comment If we denote $\arcsin x$ and $\arccos x$ by α and β respectively, we see that $\sin\alpha = x = \cos\beta$. It follows that, in the first quadrant, which applies if $x \geq 0$, α and β are complementary angles so that $\alpha + \beta = \frac{\pi}{2}$. For $-1 \leq x < 0$ the same conclusion applies although α and β lie in the 4th and 2nd quadrants respectively. In this case we have $0 < -x$ and $\arcsin x = -\arcsin(-x)$ and $\arccos x = \pi - \arccos(-x)$ so that

$$\arcsin x + \arccos x = -\arcsin(-x) + (\pi - \arccos(-x)) = \pi - (\arcsin(-x) + \arccos(-x))$$

$$= \pi - \frac{\pi}{2} = \frac{\pi}{2}.$$

6. First take $x \geq 0$, put $\theta = \cos^{-1} x$ so that (see diagram) $\sin\theta = \sqrt{1-x^2}$. If on the other hand $x < 0$, then

$$\sin(\cos^{-1} x) = \sin(\pi - \cos^{-1}(-x)) = \sin\pi\cos(\cos^{-1}(-x)) - \cos\pi\sin(\cos^{-1}(-x))$$

$$= 0 - (-\sqrt{1-(-x^2)} = \sqrt{1-x^2}.$$

$$\therefore \sin(\cos^{-1}(x)) = \sqrt{1-x^2}\,\forall\, x : |x| \leq 1.$$

7.

$$\sin\left(\sin^{-1}\left(\frac{2}{3}\right) + \cos^{-1}\left(\frac{1}{3}\right)\right) =$$

$$\sin\left(\sin^{-1}\left(\frac{2}{3}\right)\right)\cdot\cos\left(\cos^{-1}\left(\frac{1}{3}\right)\right) + \cos\left(\sin^{-1}\left(\frac{2}{3}\right)\cdot\sin\left(\cos^{-1}\left(\frac{1}{3}\right)\right) =$$

$$= \frac{2}{3}\cdot\frac{1}{3} + \sqrt{1-\frac{4}{9}}\cdot\sqrt{1-\frac{1}{9}}\,(\text{as } \cos(\sin^{-1}(x)) = \sqrt{1-x^2} = \sin(\cos^{-1} x))$$

$$= \frac{2}{9} + \frac{\sqrt{5}}{3}\cdot\frac{\sqrt{8}}{3} = \frac{2+2\sqrt{2}\cdot\sqrt{5}}{9}$$

$$= \frac{2(1+\sqrt{10})}{9}.$$

8. The quickest way to do this is first to multiply by $\sin 20°$ and then keep applying the double angle formula, $\sin x\cos x = \frac{1}{2}\sin 2x$.

With this in mind, put $a = \cos 20°\cos 40°\cos 80°$. Then, following the advice above, we obtain:

$$a\sin 20° = \sin 20°\cos 20°\cos 40°\cos 80° = \frac{1}{2}\sin 40°\cos 40°\cos 80°$$

$$= \frac{1}{4}\sin 80°\cos 80° = \frac{1}{8}\sin 160° = \frac{1}{8}\sin 20°,$$

where the final equality is justified by the general relationship, $\sin(180-x)^\circ = \sin x^\circ$. We no longer have need of the factor of $\sin 20^\circ$, and so we cancel it to obtain:

$$\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}.$$

Of course, since $\cos 60^\circ = \frac{1}{2}$ we also get

$$\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \frac{1}{16}.$$

The problem is also approachable using complex numbers, which leads to a set of more general identities that include this one as a particular case.

9. Referring to the diagram: $\tan 30^\circ = 2x \Rightarrow x = \frac{\sqrt{3}}{6}$; $z = \sin 60^\circ = \frac{\sqrt{3}}{2}$. Hence

$$\cos\alpha = \frac{x}{z} = \frac{\sqrt{3}}{6} \cdot \frac{2}{\sqrt{3}} = \frac{1}{3}.$$

10. From the viewpoint of G, *The Goblin*, the patrol boat P, comes from the SW, giving a triangle PIG, where P denotes the initial position of the Patrol Boat, G the initial position of the Goblin and I the interception point. Let us say that the interception comes after time t, so that $|PI| = 25t$ and $|GI| = 10t$. Let $\alpha = \angle GPI$. Then by the Sine Rule we have:

$$\frac{\sin\alpha}{10t} = \frac{\sin 135^\circ}{25t} \Rightarrow \sin\alpha = \frac{10}{25} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{2}{5\sqrt{2}} = \frac{\sqrt{2}}{5}.$$

We infer that

$$\theta = 45^\circ - \alpha = 45^\circ - \sin^{-1}\left(\frac{\sqrt{2}}{5}\right) = 28.57^\circ;$$

The answer to the nearest degree East of North is then 29°.

Comment It is interesting to note that the answer is independent of the initial separation of the two vessels as it is independent of the time taken for the interception.

Problem Set 5

1. From the Euler formula $e^{(u+v)i} = e^{ui}e^{vi}$ gives

$$\cos(u+v) + i\sin(u+v) = (\cos u + i\sin u)(\cos v + i\sin v)$$

$$= (\cos u\cos v - \sin u\sin v) + i(\sin u\cos v + \cos u\sin v),$$

and equating real and imaginary parts gives

$$\cos(u+v) = \cos u \cos v - \sin u \sin v, \ \sin(u+v) = \sin u \cos v + \cos u \sin v. \quad (1)$$

2. $M_{u+v} = M_u M_v$ gives

$$\begin{bmatrix} \cos(u+v) & -\sin(u+v) \\ \sin(u+v) & \cos(u+v) \end{bmatrix} = \begin{bmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{bmatrix} \begin{bmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{bmatrix}$$

$$= \begin{bmatrix} \cos u \cos v - \sin u \sin v & -\cos u \sin v - \sin u \cos v \\ \sin u \cos v + \cos u \sin v & -\sin u \sin v + \cos u \cos v \end{bmatrix}$$

and equating entries of the matrices gives the identities of Question 1.

3.

$$\begin{aligned} \cos(u-v) &= \cos(u+(-v)) = \cos u \cos(-v) - \sin u \sin(-v) \\ &= \cos u \cos v + \sin u \sin v & (2) \\ \sin(u-v) &= \sin(u+(-v)) = \sin u \cos(-v) + \cos u \sin(-v) \\ &= \sin u \cos v - \cos u \sin v & (3) \end{aligned}$$

4. Put $u = v$ in (1) we get in the first instance:

$$\cos 2u = \cos^2 u - \sin^2 u = 1 - 2\sin^2 u = 2\cos^2 u - 1 \quad (4)$$

and from the second formula in (1):

$$\sin 2u = \sin u \cos u + \cos u \sin u = 2 \sin u \cos u \quad (5)$$

5. Add the first formula in (1) to that in (2). The result is:

$$\cos(u+v) + \cos(u-v) = 2 \cos u \cos v; \quad (6)$$

Adding the second formula in (1) to that of (3):

$$\sin(u+v) + \sin(u-v) = 2 \sin u \cos v \quad (7)$$

6. Put $u + v = x$ and $u - v = y$, which is to say that $u = \frac{x+y}{2}$ and $v = \frac{x-y}{2}$ allowing us to re-write (6):

$$\cos x + \cos y = 2 \cos \big(\frac{x+y}{2}\big)\big(\frac{x-y}{2}\big)$$

Subtracting (2) from the first formula in (1) gives:

$$\cos(u+v) - \cos(u-v) = -2 \sin u \sin v$$

$$\Rightarrow \cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}.$$

7. Applying this to (7):

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}. \quad (8)$$

Finally, replace y by $-y$ in (8). We obtain:

$$\sin x + \sin(-y) = 2\sin\frac{x-y}{2}\cos\frac{x-(-y)}{2}$$
$$\Rightarrow \sin x - \sin y = 2\sin\frac{x-y}{2}\cos\frac{x+y}{2}.$$

8. Applying (1) again we obtain:

$$\tan(u+v) = \frac{\sin(u+v)}{\cos(u+v)} = \frac{\sin u\cos v + \cos u\sin v}{\cos u\cos v - \sin u\sin v} = \frac{\tan u + \tan v}{1-\tan u\tan v};$$

applying (2) and (3) we obtain:

$$\tan(u-v) = \frac{\sin(u-v)}{\cos(u-v)} = \frac{\sin u\cos v - \cos u\sin v}{\cos u\cos v + \sin u\sin v} = \frac{\tan u - \tan v}{1+\tan u\tan v}.$$

9.
$$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ\cos 30^\circ - \cos 45^\circ\sin 30^\circ$$
$$= \frac{\sqrt{2}}{2}(\frac{\sqrt{3}}{2} - \frac{1}{2}) = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

10. From (7) we obtain:

$$\int \sin 7x\cos 8x\,dx = \int \frac{1}{2}\big(\sin\big(7x+8x\big) + \sin\big(7x-8x\big)\big)\,dx$$
$$= \frac{1}{2}\int(\sin 15x - \sin x)\,dx = \frac{1}{2}\big(-\frac{1}{15}\cos 15x + \cos x\big) + c$$
$$= \frac{1}{2}\cos x - \frac{1}{30}\cos 15x + c.$$

Problem Set 6

1. Let $x \in \mathrm{dom} f$. Replacing x by $x-p$ in the definition of period we get $f(x-p) = f((x-p)+p) = f(x)$ so that $f(x) = f(x-p)\,\forall x \in \mathrm{dom} f$. In particular we may say that $f(x) = f(x \pm p)$ holds in general and by a simple induction that $f(x) = f(x+kp)$ for all $k \in \mathbb{Z}$.

2. First we observe that for all x in the domain of our function we have

$$g(x + \frac{p}{|c|}) = a + bf(c(x+\frac{p}{|c|}) + d) = a + bf(cx + d \pm p);$$

but from Question 1 we see that this also equals $a + bf(cx+d) = g(x)$. Therefore $g(x+\frac{p}{|c|}) = g(x)$ for all real x. Hence the period of $g(x)$ exists and is bounded above by $\frac{p}{|c|}$.

On the other hand, suppose that $g(x) = g(x+q)$ for all real x, where $0 < q \le \frac{p}{|c|}$. Then for all $x \in \mathbb{R}$ we have

$$a + bf(cx+d) = a + bf(c(x+q)+d)$$

$$\Rightarrow f(cx+d) = f(c(x+q)+d) \Rightarrow f(x) = f(x+cq)$$

as $b, c \ne 0$ and we may then replace x by $\frac{x-d}{c}$ throughout. Thus we obtain

$$f(x) = f(x+|c|q) \text{ by applying Question 1.}$$

It now follows by the defintiion of period that $|c|q \ge p$ so that $q \ge \frac{p}{|c|}$ and therefore we conclude that $q = \frac{p}{|c|}$, as required.

3. Suppose that q is not an integer multiple of p. Certainly $p < q$ so we may write $q = np + r$ for some remainder r with $0 < r < p$. But then for all $x \in \mathbb{R}$ we have

$$f(x+r) = f(x+q-np) = f(x+np) = f(x),$$

which, since $0 < r < p$, contradicts that p is the period of $f(x)$.

4. Since $\cos x = \sin(x - \frac{\pi}{2})$ it follows from Question 2 (with $a = 0, b = 1, c = 1$ and $d = -\frac{\pi}{2}$) that the period of $\cos x$ is the same of that of $\sin x$, which is 2π.

Next observe that $\tan(x+\pi) = \frac{\sin(x+\pi)}{\cos(x+\pi)} = \frac{-\sin x}{-\cos x} = \frac{\sin x}{\cos x} = \tan x$ for all x for which $\tan x$ is defined. On the other hand, if $\tan x = \tan(x+p)$ with $0 < p \le \pi$ then in particular we have

$$0 = \tan 0 = \tan p = \frac{\sin p}{\cos p}$$

so that $\sin p = 0$ and so $p = \pi$. Therefore the period of $\tan x$ is π.

5. By Question 2 we have the period p of $1 + 2\sec(3x - \pi)$ is $\frac{1}{3}$ of the period of $\sec x$. Since $\sec x = (\cos x)^{-1}$ and the inversion function is injective it follows that the period is the same as that of $\cos x$, whence it follows from Question 3 that $p = \frac{2\pi}{3}$.

6. First note that $\sin^2(x+\pi) = (-\sin x)^2 = \sin^2 x$ so that $\sin^2 x$ has a period of p where $0 < p \le \pi$. Next suppose that $\sin^2(x+q) = \sin^2 x$ for all real x and for some q with $0 < q \le \pi$. Then in particular $0 = \sin 0 = \sin q$, whence $\pi \le q$ and we conclude that the period of $\sin^2 x$ is indeed π.

Since the period of $\tan x$ is, by Question 4, equal to π it follows that the period p of $\tan^2 x$ exists and $0 < p \le \pi$. Then $0 = \tan^2 0 = \tan^2 p$ which implies that $\sin p = 0$ so that $p \ge \pi$. Therefore the period of $\tan^2 x$ is π. Alternatively we may argue that $\tan^2 x = 1 + \frac{1}{1-\sin^2 x}$ and since the function $g(x) = 1 + \frac{1}{1-x}$ is one-to-one, it follows that the period of $\tan^2 x$ is the same as that of $\sin^2 x$.

7. Write $3\sin x + 4\cos x = R\cos(x-\alpha)$. We get $R^2 = 3^2 + 4^2 = 5^2$so that $R = 5$ and $\tan\alpha = \frac{3}{4}$. By Questions 2 and 3 we see that the period of $5\cos(x-\alpha)$ is 2π.

8. $\sin 12x \cos 30x = \frac{1}{2}\sin 42x - \frac{1}{2}\sin 18x$. Now these terms have respective periods of $\frac{2\pi}{42} = \frac{\pi}{21}$ and $\frac{2\pi}{18} = \frac{\pi}{9}$. The required period p is therefore the least common multiple of these two periods. We require m and n with $(m,n) = 1$ such that $\frac{m}{21} = \frac{n}{9} \Rightarrow \frac{m}{n} = \frac{21}{9} = \frac{7}{3}$ and hence $p = \frac{7\pi}{21} = \frac{\pi}{3}$.

9. Let p be the period of $f(x)$. Then

$$f'(x+p) = \lim_{h\to 0}\frac{f(x+p+h) - f(x+p)}{h} = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = f'(x)$$

so that $f'(x+p) = f'(x)$ for all x and therefore $f'(x)$ is also periodic.

Comment Note that this does not prove the period of $f'(x)$ is p as there could conceivably be a smaller positive number q such that $f'(x+q) = f'(x)$.

10. Suppose that $p > 0$ and that $f(x) = \sin x^2 = \sin(x+p)^2$. Putting $x = 0$ we get $0 = \sin p^2$ so that $p^2 = k\pi$ say $(k \in \mathbb{Z}^+)$ and $p = \sqrt{k\pi}$. Now $f'(x) = 2x\cos x^2$ and, by Question 9, $f'(x) = f'(x+p)$ also. Hence

$$0 = f'(0) = f'(p) = f'(\sqrt{k\pi}) = 2p\cos(k\pi) = \pm 2p,$$

which contradicts that $p > 0$. Hence $\sin x^2$ is not a periodic function.

Problem Set 7

1. From Problem 9 on Set 4, the height of the tetrahedron h is such that

$$\frac{h}{\sqrt{3}/2} = \sin\left(\cos^{-1}\left(\frac{1}{3}\right)\right) = \sqrt{1 - \frac{1}{9}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}.$$

Hence $h = \frac{2\sqrt{2}}{3}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{3}$.

The area of the base is $\frac{1}{2}\sin\frac{\pi}{3} = \frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$. Therefore the volume of the tetrahedron is

$$V = \frac{1}{3}\times \text{base} \times \text{height} = \frac{1}{3}\frac{\sqrt{3}}{4}\cdot\frac{\sqrt{6}}{3} = \frac{3\sqrt{2}}{3^2\cdot 4} = \frac{\sqrt{2}}{12}\text{ units}^3.$$

2. (a) Substitute $y = mx + c$ in the equation of the unit circle and we find:

$x^2 + y^2 = 1$ becomes $x^2 + (mx+c)^2 - 1 = x^2 + m^2x^2 + 2mcx + c^2 - 1 = 0$

$$\Leftrightarrow (1+m^2)x^2 + (2mc)x + (c^2-1) = 0.$$

Now $y = mx + c$ is a tangent to the circle if and only if it meets the circle exactly once, which in turn is equivalent to saying that the preceding quadratic equation has a unique solution. This occurs if and only if the discriminant is zero, which is to say:

$$4m^2c^2 - 4(m^2+1)(c^2-1) = 0 \Leftrightarrow m^2c^2 = m^2c^2 - m^2 + c^2 - 1$$

$$\Leftrightarrow m^2 = c^2 - 1 \Leftrightarrow m = \pm\sqrt{c^2-1}.$$

(b) Alternatively, let us find the slope $m = m_1$ of the line AC. By noting the equality of the angles as marked on the diagram we obtain that the right triangles $\triangle ABO$ and $\triangle AOC$ are similar. By Pythagoras,

$$BC^2 + 1^2 = OC^2 \text{ and so } BC = \sqrt{c^2-1}.$$

Therefore

$$\frac{OC}{OA} = \frac{BC}{BO} \Leftrightarrow \frac{c}{a} = \frac{\sqrt{c^2-1}}{1}.$$

Since $m = c/a$ we may conclude that $m = \sqrt{c^2-1}$.

3. The minimum mirror height required to see your whole body in a mirror is half your own height. To achieve this, the top edge of the mirror should be half in-between the level of your eyes and the top of your head, and the bottom edge of the mirror should be at a level half way between your eyes and your feet. The reflecting surface must cover this much of the plane to be able to view the extremities of your body.

Surprisingly perhaps, this is independent of your distance to the mirror as can be seen by drawing a diagram that shows straight lines from your eyes going to the top of the head and to the bottom of the feet of your mirror image.

4. Locate the centre C of the cake as the intersection of the diagonals. Any line through C cuts the cakes into two (equal) halves. Next find the centre O of your circle (by finding the point of intersection of any two of its chords). The line of the cut OC cuts both the circle and the overall cake into two equal portions, as required.

5. By the Circle Theorem of Euclid, the angle $\angle ACB$ standing on a chord AB of a circle is half the angle $\angle AOB$ at the centre. In this case $\triangle AOB$ is equilateral as all of its sides are equal to the circle's radius. Hence that acute angle $\angle AOB = 60°$ so that $\angle ACB = 30°$ for any point C on the circle in the larger segment of the circle with side AB. On the other hand if C is in the smaller segement then $\angle ACB$ is half of the obtuse angle $\angle AOB = (360-60)^{\circ} = 300°$ so that $\angle ACB = 150°$.

6. Let O, C and P denote the origin, the centre of the smaller circle and let P denote the point where the circles touch. Let D be the point on the x-axis that the smaller circle touches. Since the common tangent to the circles is at right angles to the radius of each circle through P, it follows that O, C and P are collinear and $|OP| = 1$. Let r denote the radius of the smaller circle. Then by Pythagoras for the triangle ODC we obtain:

$$r^2 + r^2 = (1 - r)^2 = 1 - 2r + r^2$$

whence $r^2 + 2r - 1 = 0$. Solving and taking the positive root then gives $r = \sqrt{2} - 1$.

7. Here is just one of several possible arguments, including the use of trigonometry. Let the diagonal length be d and note that $\triangle ADB$ is similar to $\triangle ABC$ so that

$$\frac{a}{1} = \frac{1}{d} \tag{9}$$

Also $\triangle OEF$ is similar to $\triangle OBC$ so that

$$\frac{d}{1} = \frac{a}{d - 2a}$$

Hence

$$d^2 - 2ad - a = 0 \Rightarrow d^2 - 2 - \frac{1}{d} = 0 \text{ by (16).}$$

$$\text{Hence } d^3 - 2d - 1 = 0 \Rightarrow (d + 1)(d^2 - d - 1) = 0.$$

Since d is positive, it equals the positive root of $d^2 - d - 1 = 0$, which therefore gives

$$d = \frac{1 + \sqrt{1 + 4}}{2} = \frac{1 + \sqrt{5}}{2}, \text{ the Golden Ratio, } \phi.$$

Comments: The pentagon is replete with symmetries: for example $ABCF$ is a *rhombus* (square parallelogram) of unit side length, (as can now be checked). The diagonals of the pentagon meet each other in segments in the ratio $\phi : 1$ leading to an inverted copy of the pentagon appearing the vertices of which are the diagonal intersections of the parent pentagon. This kind of self-similarity behaviour is typical in mathematical objects involving ϕ.

As an interesting bonus, we can find the exact values of $\cos 36°$ and $\sin 36°$. Using the fact that the angles of a triangle sum to $180°$ and the angle at each corner of the pentagon is $\frac{360°}{5} = 108°$ we may deduce that the angle $\alpha = \angle BAC$ satisfies $2\alpha + 108° = 180°$ and so $\alpha = 36°$. Let M be the midpoint of AC and consider the right triangle $\triangle ABM$. We then obtain

$$\cos 36° = \frac{d}{2} = \frac{\phi}{2} = \frac{1 + \sqrt{5}}{4}.$$

Next using $\sin^2 \alpha = 1 - \cos^2 \alpha$ gives

$$\sin^2 36^\circ = 1 - \frac{6 + 2\sqrt{5}}{16} = \frac{10 - 2\sqrt{5}}{16}.$$

Hence

$$\sin 36^\circ = \frac{1}{4}\sqrt{10 - 2\sqrt{5}}.$$

8. BA, AC and CB are each the diagonal of a face. Hence $\triangle ABC$ is equilateral and in partciular $\angle ABC = 60^\circ$.

9. The area of an ellipse is πab where a and b are respectively the length of the sem-axes of the ellipse. From the equation we have $a^2 = 9$ and $b^2 = 49$ so that $a = 3$ and $b = 7$. Therefore the required area is $\pi \cdot 3 \cdot 7 = 21\pi$ units^2.

Comment It is easy to justify the formula for the area of an ellipse, given that a circle of radius b has area πb^2. Stretch the circle horizontally away from its vertical diameter by a factor of $\frac{a}{b}$. Imagine the circle covered by thin horizontal strips that are similarly strectched to cover the ellipse. Since each strip has its area multiplied by the factor $\frac{a}{b}$, the same applies to the ellipse (by taking the area to be the limiting value of the covering by thin strips) so the area of the ellipse is therefore $\pi b^2 \cdot \frac{a}{b} = \pi ab$. This simple result contrasts with the question of the arc length of the ellipse, which is not related to that of the circle in so simple a manner.

10. Tetrahedron: $V - E + F = 4 - 6 + 4 = 2$; Cube: $V - E + F = 8 - 12 + 6 = 2$; Octahedron: $V - E + F = 6 - 12 + 8 = 2$; Dodecahedron: $V - E + F = 20 - 30 + 12 = 2$; Icosahedron: $V - E + F = 12 - 30 + 20 = 2$.

Comment: Note that for the dual pairs (Cube, Octaderon), (Dodecahedron, Icosahedron) the vertex and face numbers are interchanged (recall Set 2 Questions 1-3). These are the five *platonic* solids and they are the only regular convex solids that can exist (*regular* meaning that each face is planar and the solid is assembled in the same manner from each vertex).

Problem Set 8

1.

$$\cos x = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = \frac{(-6 - 6 - 4)}{\sqrt{1 + 4 + 16} \cdot \sqrt{36 + 9 + 1}} = \frac{-16}{\sqrt{21} \cdot \sqrt{46}} = -0\cdot 5148 \text{ to 4 d.p.}$$

Therefore $x = 121^\circ$ (to the nearest degree).

2. Since $ABCD$ is a parallelogram

$$\mathbf{OD} = \mathbf{OA} + \mathbf{AD} = \mathbf{OA} + \mathbf{BC} = (-\mathbf{i} + 2\mathbf{j}) + (-3\mathbf{i} - 10\mathbf{j}) = -4\mathbf{i} - 8\mathbf{j}.$$

Therefore $D = (-4, -8)$.

Comment: You need to be careful to find the parallelogram $ABCD$ and not the alternative parallelograms $ABDC$ or $ACBD$. In general, given a triangle $\triangle ABC$ there are three parallelograms that may be formed with vertices A, B, and C corresponding to the three choices of sides of the triangle that act as a diagonal of the parallelogram.

3. $\mathbf{u} \cdot \mathbf{a} = 8 + 1 + 6 = 15$: $\mathbf{a} \cdot \mathbf{a} = 16 + 1 + 4 = 21$.

$$\mathbf{p} = proj_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\mathbf{a} = \frac{15}{21}\big(4\mathbf{i}-\mathbf{j}-2\mathbf{k}\big)$$
$$= \frac{20}{7}\mathbf{i}-\frac{5}{7}\mathbf{j}+\frac{10}{7}\mathbf{k}.$$

4. The vector component of u orthogonal to $\mathbf{a}$ is $\mathbf{q} = \mathbf{u} - proj_{\mathbf{a}}\mathbf{u} =$

$$(2\mathbf{i} - \mathbf{j}+3\mathbf{k}) - \big(\frac{20}{7}\mathbf{i}-\frac{5}{7}\mathbf{j}+\frac{10}{7}\mathbf{k}\big)$$
$$\mathbf{q} = -\frac{6}{7}\mathbf{i}-\frac{2}{7}\mathbf{j} + \frac{11}{7}\mathbf{k}.$$

Comment Questions 3 and 4 together furnish an example of breaking a given vector into components parallel and perpendicular to a given direction (as the outcome does not depend on the length of the second vector $\mathbf{a}$, but only its direction). This is a task that arises constantly in problems in mechanics. It is also used to solve the problem of the distance from a point to a line (see Set 10 Question 5). The formula for $\mathbf{p}$ comes from observing that $\mathbf{p} = r\hat{\mathbf{a}}$ for some $r \in \mathbb{R}$ and that $r = \mathbf{u} \bullet \hat{\mathbf{a}}$. Alternatively, we may solve more directly as $\mathbf{p} = r\mathbf{a}$, $\mathbf{p} + \mathbf{q} = \mathbf{u}$, and $\mathbf{p} \bullet \mathbf{q} = 0$. This also leads to $r = \frac{5}{7}$ and the solution.

5. If we place one corner of the cube at the origin with sides aligned to the three co-ordinate axes we then want the angle θ between the unit vector $\mathbf{i}$ and $\mathbf{d} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, which represents the vector of the diagonal of the cube from the origin. Hence we have:

$$\cos\theta = \frac{\mathbf{i} \cdot \mathbf{d}}{|\mathbf{i}| \cdot |\mathbf{d}|} = \frac{\mathbf{i} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})}{1 \cdot \sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}};$$

hence $\theta = \arccos\big(\frac{1}{\sqrt{3}}\big) \approx 54°44'$. To the nearest degree the required angle is therefore $55°$.

6. $\mathbf{r} = \mathbf{a} + \mathbf{b}t$ where $\mathbf{a} = -\mathbf{i}-2\mathbf{j}+2\mathbf{k}$ and

$$\mathbf{b} = (1 - (-1))\mathbf{i}+(7 - (-2))\mathbf{j}+(11 - 2)\mathbf{k} \Rightarrow \mathbf{b} = 2\mathbf{i}+9\mathbf{j} + 9\mathbf{k}.$$

7. From Question 6, (x, y, z) is on the line if and only if $x = -1 + 2t, y = -2 + 9t$ and $z = 2 + 9t$. Solving for t yields:

$$\frac{x+1}{2} = \frac{y+2}{9} = \frac{z-2}{9}.$$

8. Following the hint, we see that the equation has the form $5x+3y-8z+d = 0$. To determine d, we substitute the value of the included point, $(-1,-2,5)$ to get

$$5(-1)+3(-2)-8(5)+d=0 \Rightarrow -5-6-40+d=0 \Rightarrow d=51;$$

Hence the equation of the plane is $5x+3y-8z+51=0$.

9.
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 7 & 2 \\ -1 & 2 & 5 \end{vmatrix} = \mathbf{i}(35-4) - \mathbf{j}(5+2) + \mathbf{k}(2+7) = (31,-7,9).$$

10.

$$\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = \mathbf{i}(1+1) - \mathbf{j}(1-1) + \mathbf{k}(-1-1) = 2\mathbf{i} - 2\mathbf{k};$$

hence $||\mathbf{c}|| = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$ and so a required unit vector is

$$\mathbf{u} = \frac{1}{2\sqrt{2}}(2\mathbf{i} - 2\mathbf{k}) = \frac{\sqrt{2}}{2}(\mathbf{i} - \mathbf{k}).$$

Problem Set 9

1. Writing each dot product as an equation in the three unknowns x, y and z gives

$$-y+4z=-2, \; x+2y+3z=17, \; -x-y+z=-7.$$

Adding the last two gives $y+4z=10$, which when added to the first gives $8z=8 \Rightarrow z=1$, whence $y=10-4=6$, $x=17-(2\times6)-(3\times1)=17-12-3=2$. Hence $\mathbf{x}$ $=(2,6,1)$.

2. Making use of the dot product we have

$$\cos\alpha = \frac{\mathbf{v} \bullet \mathbf{i}}{|\mathbf{v}| \cdot |\mathbf{i}|} = \frac{a}{1} = a;$$

similarly the other two direction cosines can now be expressed as $\cos\beta = b$ and $\cos\gamma = c$. Finally

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = a^2+b^2+c^2 = |\mathbf{v}|^2 = 1^2 = 1.$$

3. $\overrightarrow{PQ} = ((2,1,-1)-(1,-1,0) = (1,2,-1)$, and $\overrightarrow{PR} = (-1,1,2)-(1,-1,0) = (-2,2,2)$. Hence a vector perpendicular to the plane PQR is given by $\overrightarrow{PQ} \times \overrightarrow{PR} =$

$$\det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{bmatrix}$$

$$= \mathbf{i}(2 \times 2 - ((-1) \times 2) - \mathbf{j}(1 \times 2 - (-1) \times (-2)) + \mathbf{k}(1 \times 2) - (2 \times (-2)) =$$

$$6\mathbf{i} + 6\mathbf{k}.$$

Therefore we can take $\mathbf{v} = \mathbf{i} + \mathbf{k}$ or indeed the corresponding unit vector, which is $\mathbf{v}/\sqrt{2}$. The equation of the plane then has the form $x + z = c$; substituting the point $P(1,-1,0)$ into this equation gives $1 + 0 = 1 = c$, so our equation is $x + z - 1 = 0$.

4. $\mathbf{AB} = (3,-2,2)$, $\mathbf{AC} = (-2,2,3)$ so that $\mathbf{AB} \times \mathbf{AC}$ is given by the formal determinant of:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -2 & 2 \\ -2 & 2 & 3 \end{vmatrix}$$

$$= (-6-4)\mathbf{i} - \mathbf{j}(-9+4) + \mathbf{k}(-6-4) = -10\mathbf{i} + 5\mathbf{j} - 10\mathbf{k}.$$

The required area is then

$$\frac{1}{2}|\mathbf{AB} \times \mathbf{AC}| = \frac{5}{2}\sqrt{(-2)^2 + 1^2 + (-2)^2} = \frac{5}{2}\sqrt{9} = \frac{5}{2} \cdot 3 = \frac{15}{2} \text{ units}^2.$$

5. The required distance d is given by $|(\mathbf{r_0} - \mathbf{r_1}) \bullet \mathbf{n}|$ where $\mathbf{r_0}$ is the given point, $\mathbf{r_1}$ is *any* point in the plane and $\mathbf{n}$ is a unit normal vector the plane. We have $(1,-1,4)$ is normal to the plane and has length $\sqrt{1+1+16} = \sqrt{18} = 3\sqrt{2}$. Also $\mathbf{r_0} = (2,2,2)$ and we may take $\mathbf{r_1} = (9,0,0)$. Then:

$$d = \frac{1}{3\sqrt{2}}\big|\big((2,2,2) - (9,0,0)\big) \bullet (1,-1,4)\big| = \frac{1}{3\sqrt{2}}\big|(-7,2,2) \bullet (1,-1,4)\big|$$

$$= \frac{1}{3\sqrt{2}}\big|-7-2+8\big| = \frac{1}{3\sqrt{2}} = \frac{\sqrt{2}}{6}.$$

6. The plane contains the two vectors: $\mathbf{a} = (2,4,1) - (-1,0,1) = (3,4,0)$ and $\mathbf{b} = ((8,2,-5) - (-1,0,1) = (9,2,-6)$. Hence a normal vector to our plane Π is given by

$$\mathbf{a} \times \mathbf{b} = \mathbf{i}(-24-0) - \mathbf{j}(-18-0) + \mathbf{k}(6-36) = -24\mathbf{i} + 18\mathbf{j} - 30\mathbf{k},$$

dividing by -6 we may use $\mathbf{n} = (4,-3,5)$. Hence Π has an equation of the form: $4x - 3y + 5z = c$; substituting the point $(2,4,1)$ into this equation gives $c = 8 - 12 + 5 = 1$, so Π is given by

$$4x - 3y + 5z = 1.$$

7. The vector $\mathbf{a} = (2,0,3) - (1,1,1) = (1,-1,2)$ lies in the required plane Π as does $\mathbf{b} = (1,2,-3)$ as $\mathbf{b}$ is normal to the given plane. Hence a normal to Π is given by $\mathbf{a} \times \mathbf{b} =$

$$\mathbf{i}((-1)\times(-3) - 2\times 2) - \mathbf{j}(1\times(-3) - 2\times 1) + \mathbf{k}(1\times 2 - (-1)\times 1) = -\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}.$$

Hence Π has an equation of the form $-x + 5y + 3k = c$. Substituting the point $(1,1,1)$ into this equation gives $c = -1 + 5 + 3 = 7$. Therefore the equation of Π is

$$-x + 5y + 3z = 7.$$

8. Expanding the determinant gives:

$$\mathbf{i}(u_2v_3 - u_3v_2) - \mathbf{j}(u_1v_3 - u_3v_1) + \mathbf{k}(u_1v_2 - u_2v_1)$$

$$= (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1) = \mathbf{u} \times \mathbf{v}.$$

9. The LHS when expanded gives

$$u_1(v_2w_2 - v_3u_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1)$$

which matches the expansion of the determinant.

10.

$$\begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \\ -1 & 4 & 1 \end{vmatrix}$$

$$= 1(-1-0) - (2-0) - 1(8-1) = -1 - 2 - 7 = -10,$$

and so the required volume is $|-10| = 10$.

Problem Set 10

1. Given line has vector equation $\mathbf{x} = (1,0,1) + t(3,-1,0)$ so required line is $\mathbf{x} = (3,4,1) + t(3,-1,0) \Rightarrow x = 3 + 3t,\ y = 4 - t,\ z = 1$ $(t \in \mathbb{R})$. Eliminating the parameter t we obtain:

$$\frac{x-3}{3} = 4 - y,\ z = 1.$$

2. For two lines $\mathbf{x} = \mathbf{a} + t\mathbf{b}$ and $\mathbf{y} = \mathbf{c} + s\mathbf{d}$ $(t, s \in \mathbb{R})$, the required distance is given by $d = |\mathbf{n} \bullet (\mathbf{c} - \mathbf{a})|$ where $\mathbf{n} = \frac{\mathbf{b}\times\mathbf{d}}{|\mathbf{b}\times\mathbf{d}|}$. In this case $\mathbf{a} = (3,4,1), \mathbf{b} = (3,-1,0)$, $\mathbf{c} = (0,1,3)$ and $\mathbf{d} = (-2,1,2)$. Hence $\mathbf{c} - \mathbf{a} = (-3,-3,2)$ and $\mathbf{b} \times \mathbf{d}$ is given by:

$$\mathbf{i}((-1)\times 2 - 0) - \mathbf{j}(3\times 2 - 0) + \mathbf{k}(3\times 1 - (-1)\times(-2)) = -2\mathbf{i} - 6\mathbf{j} + \mathbf{k};$$

and so $|\mathbf{b}\times\mathbf{d}| = \sqrt{4+36+1} = \sqrt{41}$. Hence

$$d = \frac{1}{9}\big|(-2,-6,1)\bullet(-3,-3,2)\big| = \frac{1}{9}\big|6+18+2\big| = \frac{26}{\sqrt{41}} = \frac{26\sqrt{41}}{41}.$$

3. A vector in the direction of the line of intersection is

$$(1,1,1)\times(2,-1,4) = \mathbf{i}(1\times4-1\times(-1)) - \mathbf{j}(1\times4-1\times2) + \mathbf{k}(1\times(-1)-1\times2)$$
$$= 5\mathbf{i}-2\mathbf{j}-3\mathbf{k}.$$

Adding the two equations gives $3x+5z=5$ so we may put $x=0, z=1$ whence $y=-x-z=-1$, giving the point $(0,-1,1)$ on the line. Hence a parametric equation for the common line is

$$\mathbf{r} = (0,-1,1)+t(5,-2,-3) = (5t,\ -1-2t,\ 1-3t).$$

4. From Question 1 we get $\mathbf{r}(t) = (x(t),y(t),z(t)) = (5t,1-2t,-1-3t)$. Making t the subject of each co-ordinate formula we obtain:

$$\frac{x}{5} = \frac{y+1}{-2} = \frac{z-1}{-3}.$$

5. First find a vector $\mathbf{u}$ to the given point $(2,0,-3)$ from a point on the line: we put $t=0$ to get $\mathbf{i}+\mathbf{j}-3\mathbf{k} = (1,1,-3)$, so that $\mathbf{u} = (1,-1,0)$. Next project $\mathbf{u}$ onto the direction of the line, which is $\mathbf{a} = (0,3,4)$ as $\mathbf{r}(t) = (1,1,-3)+t(0,3,4)$. We obtain

$$\mathbf{p} = \frac{\mathbf{u}\bullet\mathbf{a}}{\mathbf{a}\bullet\mathbf{a}}\mathbf{a} = \frac{(0-3+0)}{0+9+16}\mathbf{a} = -\frac{3}{25}\mathbf{a}.$$

Hence the normal vector from the line to our point is $\mathbf{n} = \mathbf{u}-\mathbf{p} = (1,-1,0)+\frac{3}{25}(0,3,4) = (1,-\frac{16}{25},\frac{12}{25})$. The required distance is then

$$||\mathbf{n}|| = (1+\frac{256+144}{625})^{\frac{1}{2}} = \big(\frac{1025}{625}\big)^{\frac{1}{2}} = \frac{\sqrt{41}}{5}.$$

6. The area of the rotated circle is πr^2 and its centroid is its centre, which travels a distance $2\pi d$ to generate the torus. By Pappus, the volume of revolution, V is thus

$$V = (\pi r^2)(2\pi d) = 2\pi^2 r^2 d.$$

7. The area of the semicircle is $\frac{\pi}{2}r^2$ and its centroid travels a distance $2\pi d$, where d is the distance from the centre of the circle along the line at right angles to its diameter to the centroid, C, of the solid semicircle.

The volume generated is that of a sphere of radius r, which is thus $\frac{4}{3}\pi r^3$. Hence Pappus gives the equation:

$$\frac{4}{3}\pi r^3 = \big(\frac{1}{2}\pi r^2\big)(2\pi d) = \pi^2 r^2 d \text{ and so } d = \frac{4\pi r^3}{3\pi^2 r^2} = \frac{4r}{3\pi}.$$

8. The perimeter of the generating circle has length $2\pi r$ and its centroid is the centre of the circle, which travels a distance $2\pi d$ in sweeping out the torus. Hence the surface area S is:

$$S = (2\pi r)(2\pi d) = 4\pi^2 rd.$$

9. Rotating the wire about its diameter generates a sphere of surface area $A = 4\pi r^2$. By symmetry, the centroid of the wire lies on the radius at right angles to this diameter at a distance d say from the centre. Since the perimeter of the generating curve is πr, we have by Pappus that the surface area of the sphere equals $A = (2\pi d)(\pi r)$. Equating these two expressions for S gives us:

$$A = 4\pi r^2 = 2\pi^2 rd \text{ and so } d = \frac{4\pi r^2}{2\pi^2 r} = \frac{2r}{\pi}.$$

10. Let s denote the slant height of the cone. The surface of the cone is generated by rotating the slanting edge around the axis of the cone. The centroid of the line segment generator is the midpoint of the slant and is a distance $\frac{r}{2}$ from the axis of rotation. Hence, by Pappus, the surface area A of the cone is $A = 2\pi\left(\frac{r}{2}\right)s = \pi rs$. By Pythagoras, we have $s^2 = r^2 + h^2$ so that in terms of r and h the surface area of the cone is given by:

$$S = \pi r\sqrt{r^2 + h^2}.$$