Mathematics 204 Complex Variables Solutions

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This module builds on your existing knowledge of complex numbers to begin the study of functions of a complex variable, which holds many surprises. The first two problem sets continue the theme with the introduction of *stereographic projection* and problems involving the standard tool of using the complex exponentiation function and moving to real and imaginary parts. A complex variable can be viewed as a single variable and so the definition of differentiability of a real function extends to a complex one. However, at the same time it partakes of the nature of two variable functions in that the limit must exist through all directions of approach and the result is that *complex differentiablity* is very demanding in that the real and imaginary parts of the function must be linked through the *Cauchy-Riemann equations*, which are the subject of Set 3.

Contour integration, which the student will have seen in the context of vector functions, is introduced in Set 4 but the special nature of integration in the complex plane is explored through the *Cauchy integration formula* of set 5.

Since all differentiable functions of a complex variable are analytic and can be represented by series, the topic of series arise often in the problem sets, including Set 6, where the *complex logarithm* function is also introduced. In Set 7 we study functions that are not analytic but can be represented by series that allows for negative powers of the complex variable z, the so-called *Laurent series*. In Set 8 the emphasis is on the *Cauchy Residue theorem* and its application in calculating integrals, including sometimes results for integrals along the real line. In Set 9 there is a variety of further problems making use of the techniques that have been introduced while Set 10 introduces the celebrated *Riemann zeta function* and some of its remarkable properties are to be found there.

Solutions and Comments for the Problems

Problem Set 1

1. Using the $|z|^2 = z\overline{z}$ and the additive and multiplication properties of conjugation the LHS can be written as

$$(z_1 + z_2)(\bar{z}_1 + \bar{z}_2) + (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$$

$$= z_1\bar{z}_1 + z_2\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_2 + z_1\bar{z}_1 - z_1\bar{z}_2 - z_2\bar{z}_1 + z_2\bar{z}_2$$
$$= 2(z_1\bar{z}_1 + z_2\bar{z}_2) = 2(|z_1|^2 + |z_2|^2).$$

2. Write $z = x + iy$ and the equation becomes

$$az+\bar{a}\bar{z}+b = a(x+iy)+\bar{a}(x-iy)+ih = (a-\bar{a})x+\bar{a}(x-iy)+ih = (a-\bar{a})x+i(a+\bar{a})y+ih$$

$$= 2i\mathrm{Im}(a)x + 2i\mathrm{Re}(a) + ih = 0$$
$$\Rightarrow Ax + By + C = 0$$

where $A = 2\mathrm{Im}(a)$, $B = 2\mathrm{Re}(a)$ and $C = h$.

3. In general the technique is to write $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$ and simplify. Here we see however

$$f(z) = 2x(1 - y) + i(x^2 - y^2 + 2y)$$

$$= 2x + 2yi + i(x^2 - y^2 + 2xyi)$$
$$= 2(x + iy) + i(x + iy)^2$$
$$\therefore f(z) = iz^2 + 2z.$$

4.
$$(1 + i)(x + iy) = (x - y) + i(x + y)$$

so $\mathrm{Re}((1 + i)z) = x - y > 0$, which is to say $y < x$. The region is the half plane in the complex plane strictly below the line $y = x$.

5. The line L runs between $N = (0, 0, 1)$ and $w = (x, y, 0)$ so that a typical point on L has the form

$$(x, y, 0) + t((0, 0, 1) - (x, y, 0))$$

$$= \{((1 - t)x, (1 - t)y, t), \ -\infty < t < \infty\}.$$

6. The coordinates of W in terms of t therefore satisfy

$$1 = (1 - t)^2x^2 + (1 - t)^2y^2 + t^2 = (1 - t)^2|w|^2 + t^2$$

$$\Rightarrow 1 - t^2 = (1 - t)^2|w|^2.$$

Since $t \neq 0$ as $|w| \neq \infty$ we arrive at

$$t^2(|w|^2+1) - 2t|w|^2 + |w|^2 - 1 = 0$$

$$\Rightarrow t = \frac{2|w|^2 \pm \sqrt{4|w|^4 - 4|w|^4 + 4}}{2(|w|^2+1)} = \frac{|w|^2-1}{|w|^2+1}.$$

Since

$$1 - t = 1 - \frac{|w|^2-1}{|w|^2+1} = \frac{2}{|w|^2+1},$$

we find that the coordinates of W for $w = x + iy$ are therefore

$$W = (x_1, x_2, x_3) = \Big(\frac{2x}{|w|^2+1}, \frac{2y}{|w|^2+1}, \frac{|w|^2-1}{|w|^2+1}\Big). \tag{1}$$

7. Given $W = (x_1, x_2, x_3)$ we may find w by setting $t = x_3$ in the equation of L to get

$$w = x + iy = \frac{x_1}{1-x_3} + \frac{ix_2}{1-x_3} = \frac{x_1 + ix_2}{1-x_3}.$$

8.

$$d^2(W, W') = (x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2$$
$$= (x_1^2 + x_2^2 + x_3^2) + (x_1'^2 + x_2'^2 + x_3'^2) - 2x_1x_1' - 2x_2x_2' - 2x_3x_3'$$
$$= 2 - 2(x_1x_1' + x_2x_2' + x_3x_3').$$

9. We may write (1) also as

$$W = \Big(\frac{w+\bar{w}}{|w|^2+1}, \frac{i(\bar{w}-w)}{|w|^2+1}, \frac{|w|^2-1}{|w|^2+1}\Big)$$

Continuing the calculation:

$$d^2(W, W') = 2 - 2\frac{(w+\bar{w})(w'+\bar{w'}) + (w-\bar{w})(\bar{w'}-w') + (|w|^2-1)(|w'|^2-1)}{(|w|^2+1)(|w'|^2+1)}$$

$$= 2\Big(\frac{|w|^2 + |w'|^2 + |ww'|^2 + 1 - ww' - w\bar{w'} - \bar{w}w' - \overline{ww'} - \cdots}{(1+|w|^2)(1+|w'|^2)}\Big)$$

$$= 2\Big(\frac{\cdots w\bar{w'} + ww' + \overline{ww'} - \bar{w}w' - |ww'|^2 + |w|^2 + |w'|^2 - 1}{(1+|w|^2)(1+|w'|^2)}\Big)$$

$$= \frac{4(|w|^2 + |w'|^2 - w\bar{w'} - \bar{w}w')}{(1+|w|^2)(1+|w'|^2)} = \frac{4(w\bar{w} + w'\bar{w'} - w\bar{w'} - \bar{w}w')}{(1+|w|^2)(1+|w'|^2)}$$

$$= \frac{4(w-w')(\bar{w}-\bar{w'})}{(1+|w|^2)(1+|w'|^2)} = \frac{4|w-w'|^2}{(1+|w|^2)(1+|w'|^2)}$$

$$\Rightarrow d(W, W') = \frac{2|w-w'|}{[(1+|w|^2)(1+|w'|^2)]^{\frac{1}{2}}}.$$

10. As $|w'| \to \infty$we have $W' \Rightarrow N = (0,0,1)$ and so

$$d^2(W,N) = x_1^2 + x_2^2 + (x_3-1)^2 = x_1^2 + x_2^2 + x_3^2 - 2x_3 + 1 = 2(1-x_3);$$

$$\text{Now } 1 - x_3 = 1 - \frac{|w|^2-1}{|w|^2+1} = \frac{2}{1+|w|^2}$$

$$\Rightarrow d(W,N) = \frac{2}{(1+|w|^2)^{\frac{1}{2}}}.$$

Problem Set 2

1. The roots of $z^n - 1 = 0$ have the form $e^{\frac{2\pi i}{n}} = e^{\frac{i\pi}{10}}$ so the least value of n for which this is true satisfies $\frac{2}{n} = \frac{1}{10}$ so that $n = 20$.

2. Now for $z = x + iy$

$$\cos z = \cos x \cosh y - i \sin x \sinh y = \frac{1}{\sqrt{2}}.$$

For $\mathrm{Im}(z) = 0$ we have either that $\sinh y = 0 \Leftrightarrow y = 0$ in which case$\cosh(0) = 1$, $\cos x = \frac{1}{\sqrt{2}} \Leftrightarrow x = 2n\pi \pm \frac{\pi}{4}$ or $\sin x = 0$, so that $\cos x = \pm 1$, $\cosh y = \mp \frac{1}{\sqrt{2}}$. However, since $\cosh y \geq 1$ for all y, there are no such solutions. Hence the solution set is

$$\{z = x + iy \in \mathbb{C} : x = 2n\pi \pm \frac{\pi}{4}, y = 0,\, n \in \mathbb{Z}\}.$$

3. Let $z = x + iy$ so that

$$\mathbb{R}(\frac{1}{z}) = \mathbb{R}(\frac{x-iy}{x^2+y^2}) = \frac{x}{x^2+y^2} = c$$

$$\Rightarrow x^2 - \frac{x}{c} + y^2 = 0$$

$$\Rightarrow (x - \frac{1}{2c})^2 + y^2 = \frac{1}{4c^2},$$

which is the equation of a circle centred at $(\frac{1}{2c}, 0)$ with radius $\frac{1}{2c}$.

4 & 5. Put $z = e^{\frac{2i\pi}{n}} = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}$. Then

$$\sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z} = \frac{1-1}{1-z} = 0.$$

Since $z^k = e^{\frac{2ik\pi}{n}} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$ we obtain by first taking first the real, and then the imaginary parts of the previous equality that

$$\sum_{k=0}^{n-1} \cos\frac{2k\pi}{n} = 0 \Rightarrow \sum_{k=1}^{n-1} \cos\frac{2k\pi}{n} = -1, \text{ and}$$

$$\sum_{k=0}^{n-1} \sin\frac{2k\pi}{n} = 0 \Rightarrow \sum_{k=1}^{n-1} \sin\frac{2k\pi}{n} = 0.$$

6.

$$z^n = (1+z)^n \Rightarrow \big(\frac{z+1}{z}\big)^n = 1 \Rightarrow 1 + \frac{1}{z} = e^{\frac{2ik\pi}{n}}$$

$$\Rightarrow z = \frac{1}{e^{\frac{2ik\pi}{n}} - 1} \ (1 \le k \le n-1)$$

$$\Rightarrow z = \frac{1}{e^{\frac{ik\pi}{n}}(e^{\frac{ik\pi}{n}} - e^{-\frac{ik\pi}{n}})} = \frac{e^{-\frac{ik\pi}{n}}}{2i\sin k\pi} = \frac{\cos\frac{k\pi}{n} - i\sin\frac{k\pi}{n}}{2i\sin\frac{k\pi}{n}}$$

$$= \frac{\sin\frac{k\pi}{n} + i\cos\frac{k\pi}{n}}{-2\sin\frac{k\pi}{n}} = -\frac{1}{2}\big(1 + i\cot\frac{k\pi}{n}\big);$$

and in particular $\mathrm{Re}(z) = -\frac{1}{2}$, so all $n-1$ solutions lie on that line.

7. Following the hint we note that for $z = x + iy$ that $z + \overline{z} = 2\mathbb{R}(z)$ and since $x^2 \le x^2 + y^2$ and $|z| \ge 0$ it follows from $\mathbb{R}(z)^2 \le |z|^2$ that $\mathbb{R}(z) \le |z|$. Now replacing z by $z\overline{w}$ in the equality $z + \overline{z} = 2\mathbb{R}(z)$ we obtain

$$z\overline{w} + \overline{z\overline{w}} = 2\mathbb{R}(z\overline{w})$$

$$\Rightarrow \frac{z\overline{w} + \overline{z}w}{2} = \mathbb{R}(z\overline{w}) \le |z\overline{w}| = |z||\overline{w}| = |z||w|.$$

8.

$$(z+w)^2 = (z+w)\overline{(z+w)} = (z+w)(\overline{z} + \overline{w})$$

$$= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \le |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2;$$

where the inequality comes from Question 7. Taking square roots of both sides then gives the Triangle Inequality:

$$|z + w| \le |z| + |w|.$$

9. Let us write

$$\mu = \frac{z_3 - z_2}{z_2 - z_1} \text{ then}$$

$$z_3 - z_2 = \mu(z_2 - z_1) \tag{2}$$

$$z_3 - z_1 = z_3 - z_2 + z_2 - z_1 = (1+\mu)(z_2 - z_1) \tag{3}$$

Substituting (2) and (3) in the given equality and cancelling the common factor of $|z_2 - z_1|^2$ gives

$$|1 + \mu|^2 = 1 + |\mu|^2$$

$$\Rightarrow (1+\mu)(1+\overline{\mu}) = 1 + \mu\overline{\mu} \Rightarrow \mu + \overline{\mu} = 2\mathbb{R}(\mu) = 0.$$

Therefore $\mu = i\beta$ for some $\beta \in \mathbb{R}$, which is the result required.

10. Let $z = x + ic$ where $c \in \mathbb{R}$ is a constant, so that z represents a line parallel to the real axis. Then

$$\sin z = \sin(x + ic) = \sin x \cosh c + i \cos x \sinh c.$$

Write $u = \sin x(\cosh c)$ and $v = \cos x(\sinh c)$. Then

$$\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1;$$

giving an ellipse in the uv-plane, centred at the origin with semi-minor axis lengths of $\cosh c$ in the u-direction and $|\sinh c|$ in the v-direction.

For a line parallel to the Imaginary axis we take $z = c + iy$ so we have

$$\sin z = \sin(c + iy) = \sin c \cosh y + i \cos c \sinh y,$$

writing $u = (\sin c)\cosh y$ and $v = (\cos c)\sinh y$ we obtain:

$$\frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c} = 1;$$

which is the equation of a rectangular hyperbola centred at the origin with asymptotes given by $v = \pm(\cot c)u$.

Problem Set 3

1.

$$f(z) = z^2 = (x + iy)^2 = (x^2 - y^2) + 2xyi$$
$$\Rightarrow u(x, y) = x^2 - y^2,\ v(x, y) = 2xy.$$

We have

$$u(x, y) = x^2 - y^2 \Rightarrow \frac{\partial u}{\partial x} = 2x,\ \frac{\partial u}{\partial y} = -2y,\ v(x, y) = 2xy \Rightarrow \frac{\partial v}{\partial x} = 2y,\ \frac{\partial v}{\partial y} = 2x.$$

$$\therefore\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x;\ \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$
$$\Rightarrow f'(z) = 2x + 2yi = 2(x + iy) = 2z.$$

2.

$$f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos x + i \sin y)$$
$$\Rightarrow u(x, y) = e^x \cos y,\ v(x, y) = e^x \sin y.$$

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$$

$$\Rightarrow f'(z) = e^x \cos y + ie^x \sin y = e^z.$$

3.

$$f(z) = \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

$$\Rightarrow u(x,y) = \frac{x}{x^2+y^2},\ v(x,y) = -\frac{y}{x^2+y^2}.$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)(1)-(2x)x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)(-1)-(-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial u}{\partial x}.$$

$$\frac{\partial u}{\partial y} = \frac{(0)(x^2+y^2)-2y(x)}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = -\frac{(0)(x^2+y^2)-(2x)y}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2} = -\frac{\partial u}{\partial y}.$$

$$\Rightarrow f'(z) = \frac{y^2-x^2}{(x^2+y^2)^2} - i\frac{2xy}{(x^2+y^2)^2} = -\frac{(x-iy)^2}{((x+iy)(x-iy))^2}$$

$$= -\frac{1}{(x+iy)^2} = -\frac{1}{z^2}.$$

Hence for $f(z) = z^{-1}$ we have $f'(z) = -z^{-2}$.

4.

$$u(x,y) = \frac{1}{2}\ln(x^2+y^2),\ v(x,y) = \arctan\big(\frac{y}{x}\big)$$

$$u_x = \frac{x}{x^2+y^2},\ v_y = \frac{(1/x)}{1+(y/x)^2} = \frac{x}{x^2+y^2} = u_x;$$

$$u_y = \frac{y}{x^2+y^2},\ -v_x = -\frac{(-y/x^2)}{1+(y/x)^2} = \frac{y}{x^2+y^2} = u_y$$

$$\Rightarrow f'(z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$$= \frac{1}{2}\big(\frac{z+\overline{z}}{z\overline{z}}\big) - i\big(\frac{z-\overline{z}}{iz\overline{z}}\big)\big) = \frac{1}{2}\big(\frac{1}{\overline{z}} + \frac{1}{z} - \frac{1}{\overline{z}} + \frac{1}{z}\big) = \frac{1}{z}.$$

Comment Indeed Log$z$ is differentiates to $\frac{1}{z}$ everwhere away from the *slit* of the non-positive reals.

5. $f(z) = \overline{z} = x - iy$ so that $u(x,y) = x$ and $v(x,y) = -y$. Then $\frac{\partial u}{\partial x} = 1$ while $\frac{\partial v}{\partial y} = -1$, so the first equation is violated and so $f(z) = \overline{z}$ is not differentiable. (However, the second CR equation holds here as $u_y = v_x = -v_x = 0$).

Similarly $g(z) = \sqrt{x^2+y^2} = u(x,y)$ while $v(x,y) = 0$: clearly both CR equations fail.

6. We have by the Cauchy Riemann equations that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\Big(\frac{\partial v}{\partial y}\Big) = \frac{\partial^2 v}{\partial x \partial y} \text{ and } \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\Big(-\frac{\partial v}{\partial x}\Big) = -\frac{\partial^2 v}{\partial y \partial x}$$

and so by equality of mixed partial derivatives, the sum of these two terms vanishes.

7. First

$$\frac{\partial u}{\partial x} = 2(1-y),\ \frac{\partial^2 u}{\partial x^2} = 0,\ \frac{\partial u}{\partial y} = -2x\ \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 + 0 = 0.$$

Hence $u(x, y)$ is harmonic. We require $v(x, y)$ to satisfy the Cauchy-Riemann equations so we put

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2 - 2y \Rightarrow v(x, y) = 2y - y^2 + \phi(x)$$

for some $\phi(x)$ to be determined. Then by the second equation we obtain:

$$\frac{\partial v}{\partial x} = \phi'(x) = -\frac{\partial u}{\partial y} = -(-2x) = 2x \Rightarrow \phi(x) = x^2 + c.$$

$$\therefore\ v(x.y) = x^2 - y^2 + 2y + c.$$

8.

$$u(x, y) = 2y^3 - 6x^2y + 4x^2 - 7xy - 4y^2 + 3x + 4y - 4$$
$$\Rightarrow u_x = -12xy + 8x + 7y + 3 = v_y$$
$$\Rightarrow v(x, y) = -6xy^2 + 8xy + \frac{7}{2}y^2 + 3y + \phi(x)$$
$$\Rightarrow -v_x = 6y^2 - 8y - \phi'(x) = u_y = 6y^2 - 6x^2 - 7x - 8y + 4$$
$$\Rightarrow \phi'(x) = 6x^2 + 7x + 4 \Rightarrow \phi(x) = 2x^3 + \frac{7}{2}x^2 + 4x + c.$$
$$\therefore v(x, y) = -6xy^2 + 8xy + \frac{7}{2}y^2 + 3y + 2x^3 + \frac{7}{2}x^2 + 4x + c\ (c \in \mathbb{R}).$$

9.

$$u(x, y) = \sinh x \sin y \Rightarrow u_x = \cosh x \sin y = v_y$$

$$\Rightarrow v(x, y) = -\cosh x \cos y + \phi(x) \Rightarrow -v_x = \sinh x \cos y - \phi'(x) = u_y = \sinh x \cos y.$$

Hence $\phi(x) = c \in \mathbb{R}$ and so

$$v(x, y) = -\cosh x \cos y + c$$

$$\Rightarrow f(z) = \sinh x \sin y - i \cosh x \cos y + ic,\ (c \in \mathbb{R}).$$

Now

$$\cosh z = \frac{e^z + e^{-z}}{2} = \frac{e^x e^{iy} + e^{-x}e^{-iy}}{2} = \frac{e^x(\cos y + i \sin y) + e^{-x}(\cos y - i \sin y)}{2}$$

$$= \cos y\left(\frac{e^x + e^{-x}}{2}\right) + i \sin y\left(\frac{e^x - e^{-x}}{2}\right) = \cos y \cosh x + i \sinh x \sinh y$$
$$= i(\sinh x \sin y - i \cosh x \cos y) = if(z)$$
$$\therefore f(z) = -i \cosh z.$$

10. $z = re^{i\theta} = r\cos\theta + ir\sin\theta$. Hence $u(r,\theta) = r\cos\theta$ and $v(r,\theta) = r\sin\theta$. Then

$$\frac{\partial u}{\partial r} = \cos\theta,\ \frac{\partial v}{\partial \theta} = r\cos\theta \Rightarrow \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}\cdot r\cos\theta = \cos\theta = \frac{\partial u}{\partial r}.$$

$$\frac{\partial v}{\partial r} = \sin\theta,\ \frac{\partial u}{\partial \theta} = -r\sin\theta \Rightarrow -\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{1}{r}(-r\sin\theta) = \sin\theta = \frac{\partial v}{\partial r}.$$

Problem Set 4

1. On C we have $|z| = 1$. Using the standard parametrization $z = e^{it}$ $(0 \le t \le 2\pi)$ we get $dz = ie^{it}\,dt$ and so

$$\int_C |z|\,dz = \int_0^{2\pi} 1\cdot ie^{it}\,dt = [e^{it}]_0^{2\pi} = e^{2\pi i} - 1 = 0.$$

2. On C we have $\mathrm{Re}(z) = \cos t = \frac{1}{2}(e^{it} - e^{-it})$ so we obtain:

$$\int_C \mathrm{Re}(z)\,dz = \tfrac{i}{2}\textstyle\int_0^{2\pi}(e^{it}+e^{-it})e^{it}\,dt = \tfrac{i}{2}\int_0^{2\pi}(e^{2it}+1)\,dt$$

$$= \frac{i}{2}[\frac{e^{2it}}{2i} + t]_0^{2\pi} = \frac{i}{2}[\frac{e^{4\pi i}}{2i} + 2\pi - \frac{1}{2i} - 0] = \frac{1}{4} + \pi i - \frac{1}{4} = \pi i.$$

3. On C we have $\overline{z} = e^{-it}$ so we obtain:

$$\int_C \overline{z}\,dz = i\int_0^{2\pi} e^{-it}e^{it}\,dt = 2\pi i.$$

4. Parametrize the circle by $z(t) = z_0 + \rho e^{it}$ $(0 \le t \le 2\pi)$. Then $\frac{dz}{dt} = i\rho e^{it}$ and the integral becomes

$$\int_0^{2\pi} (\rho e^{ut})^n \cdot i\rho e^{it}\,dt = i\rho^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,dt$$

$$= i\rho^{n+1}\Big[\frac{1}{i(n+1)}e^{int}\Big]_0^{2\pi} = \frac{\rho^{n+1}}{n+1}(e^{2i\pi} - e^0) = 0.$$

5. If on the other hand $n = -1$ in the previous problem our integral becomes

$$\int_0^{2\pi} i\,dt = 2\pi i.$$

6. Our parametrization this time is $z(t) = e^{it}$ $(0 \le t \le \frac{\pi}{2})$. On C, $|z| = 1$ so we get

$$\int_0^{\frac{\pi}{2}} ie^{it}\,dt = \big[e^{it}\big]_0^{\frac{\pi}{2}} = (\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) - (\cos 0 + i\sin 0)$$

$$= (0 + i) - (1 + 0) = i - 1.$$

7. Parametrizing the two line segments separately: in the first segment $z(t) = 1 - t$, $(0 \leq t \leq 1)$ so that $dz = -dt$ and in the second $z(t) = it$ $(0 \leq t \leq 1)$ so $dz = idt$. Hence we obtain:

$$\int_C |z|\, dz = -\int_0^1 |1 - t|\, dt + i \int_0^1 |t| dt$$

$$= \Big[\frac{(1-t)^2}{2} + i\frac{t^2}{2}\Big]_0^1 = [(0 + \frac{i}{2}) - (\frac{1}{2} + 0) = \frac{i - 1}{2}.$$

8. Integrand has an anti-derivative in $F(z) = -z\cos z + \sin z$ and the contour runs from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ so that

$$I = [-\frac{\pi}{2}\cos(\frac{\pi}{2}) + \sin(\frac{\pi}{2}) - (-\frac{\pi}{2}\cos(-\frac{\pi}{2}) + \sin(-\frac{\pi}{2})]$$

$$= 1 + 1 = 2.$$

9. Since $z^2 + z = z(z + 1)$ we use partial fractions to write the integrand as $\frac{A}{z} + \frac{B}{z+1}$. By the *cover-up* method we get on putting $z = 0$ in $\frac{3z+5}{z+1}$ gives that $A = \frac{5}{0+1} = 5$ and putting $z = -1$ in $\frac{3z+5}{z}$ gives $B = \frac{3(-1)+5}{-1} = -2$. Hence our integrand $f(z) = \frac{5}{z} - \frac{2}{z+1}$. This is analytic everywhere except the first term $f_1(z) = \frac{5}{z}$ is not defined at $z = 0$ and the second $f_2(z) = -\frac{2}{z+1}$ at $z = -1$. Using the result of Question 5 and the *Principle of Deformation* we now obtain:

$$\int_C \frac{5\, dz}{z} = 5 \int_{|z|=1} \frac{dz}{z} = 5 \cdot 2\pi = 10\pi i.$$

Similarly

$$\int_C \frac{2\, dz}{z+1} = 2 \int_{|z+1|=1} \frac{dz}{z} = 2 \cdot 2\pi = 4\pi i$$

$$\int_C \frac{3z+5}{z^2+z}\, dz = 10\pi i - 4\pi i = 6\pi i.$$

10. An anti-derivative of $z \sinh z^2$ is $\frac{1}{2}\cosh z^2$ so we get

$$\frac{1}{2}\big[\cosh z^2\big]_i^{3i} = \frac{1}{2}\big[(\cosh(-9) - \cosh(-1)\big] = \frac{\cosh 9 - \cosh 1}{2}.$$

Problem Set 5

1. $\frac{\cos z}{z}$ fails to by analytic at $z = 0$, which is inside the circle. Put $f(z) = \cos z$ and $z_0 = 0$. By the Cauchy formula we then obtain:

$$f(0) = \frac{1}{2\pi i} \int_C \cos z \cdot \frac{dz}{z - 0}$$

$$\Rightarrow \int_C \frac{\cos z}{z}\,dz = 2\pi i \cdot \cos 0 = 2\pi i.$$

2. The base $n = 0$ case is the Cauchy integration formula. Let $n \geq 1$. Then

$$f^{(n)}(z_0) = (f^{(n-1)}(z_0))' = \big(\frac{(n-1)!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^n}\,dz\big)'$$

$$= \frac{(n-1)!}{2\pi i}\oint_C \frac{d}{dz_0}\big(\frac{f(z)}{(z-z_0)^n}\big)\,dz = \frac{(n-1)!}{2\pi i}\oint_C \frac{(-1)(-n)f(z)}{(z-z_0)^{n+1}}\,dz$$

$$= \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz,$$

and so the induction continues, are required.

3. Note that $\frac{z^2}{z^2+1} = \frac{z^2}{(z+i)(z-i)}$, which is defined and analytic everywhere except $z = \pm i$. For the first circle, only $z = -i$ is in the circle $|z+i| = 1$ so set $f(z) = \frac{z^2}{z-i}$ and $z_0 = -i$. The Cauchy formula then gives:

$$f(-i) = \frac{1}{2\pi i}\int_{|z+i|=1} \frac{z^2}{z-i}\cdot\frac{dz}{z+i}$$

$$\Rightarrow \int_{|z+i|=1} \frac{z^2\,dz}{z^2+1} = 2\pi i \cdot \frac{(-i)^2}{-i-i} = \frac{-2\pi i}{-2i} = \pi.$$

4. This time only $z = i$ lies inside or on the circle $|z-i| = \frac{1}{2}$ so put $f(z) = \frac{z^2}{z+i}$ and $z_0 = i$. By the formula we get:

$$f(i) = \frac{1}{2\pi i}\int_{|z-i|=\frac{1}{2}} \frac{z^2}{z+i}\cdot\frac{dz}{z-i}$$

$$\Rightarrow \int_{|z-i|=\frac{1}{2}} \frac{z^2\,dz}{z^2+1} = 2\pi i \cdot \frac{i^2}{i+i} = \frac{-2\pi i}{2i} = -\pi.$$

5. We may re-write the integral as

$$\int_{|z|=1} \frac{\frac{1}{4}z^2}{(z-\frac{1}{2})^2}\,dz.$$

Now $z = \frac{1}{2}$ lies within the circle C and the integral is in the form of the derivatives formula, with $f(z) = \frac{z^2}{4}$ and $n = 1$, $z_0 = \frac{1}{2}$. Applying that formula then gives

$$f'(\frac{1}{2}) = \frac{1}{2\pi i}\int_{|z|=1} \frac{\frac{z^2}{4}}{(z-\frac{1}{2})^2}\,dz.$$

Now $f'(z) = \frac{z}{2}$ so that $f'(\frac{1}{2}) = \frac{1}{4}$. Hence

$$\int_{|z|=1} \frac{z^2}{(2z-1)^2}\,dz = \frac{2\pi i}{1!}f'(\frac{1}{2}) = \frac{2\pi i}{4} = \frac{\pi i}{2}.$$

6. We have $z = 0$ lies within the circle so set $f(z) = \cos z$, $n = 1$ and $z_0 = 0$. Then $f'(z) = -\sin z$ and by the derivatives theorem:

$$\int_{|z|=1} \frac{\cos z}{z}\, dz = \frac{2\pi i}{1!} \cdot f'(0) = -2\pi i \cdot \sin 0 = 0.$$

7. Again $z = 0$ lies in the circle. This time set $f(z) = e^{z^3}$, $n = 2$ and $z_0 = 0$. Then $f'(z) = 3z^2 e^{z^3}$ and $f''(z) = 6ze^{z^3} + 9z^4 e^{z^3} = 3e^{z^3}(2z + 3z^4)$. By the derivatives theorem we obtain:

$$\int_{|z|=1} \frac{e^{z^3}}{z^3}\, dz = \frac{2\pi i}{2!} \cdot 3e^{z^3}(2z + 3z^4)_{z=0} = 0.$$

8 & 9. Since $w = z+1$ we have $z = w-1$ and so $4-6z = 4-6(w-1) = 10-6w$ and $2z^2 - 3z + 1 = 2(w-1)^2 - 3(w-1) + 1 = 2w^2 - 4w + 2 - 3w + 3 + 1 = 2w^2 - 7w + 6 = (2w-3)(w-2)$. Hence

$$\frac{4-6z}{2z^2-3z+1} = \frac{10-6w}{(2w-3)(w-2)} = \frac{5-3w}{(w-\frac{3}{2})(w-2)} \equiv \frac{A}{w-\frac{3}{2}} + \frac{B}{w-2};$$

by the cover-up rule

$$A = \frac{5-\frac{9}{2}}{\frac{3}{2}-2} = -1,\ B = \frac{5-6}{2-\frac{3}{2}} = -2, \text{ so}$$

$$\Rightarrow \frac{5-3w}{(w-\frac{3}{2})(w-2)} = \frac{1}{\frac{3}{2}-w} + \frac{2}{2-w} = \frac{\frac{2}{3}}{1-\frac{2}{3}w} + \frac{1}{1-\frac{w}{2}}.$$

Hence for $|w| < \frac{3}{2}$ we have $\frac{1}{1-\frac{2}{3}w} = 1 + \frac{2}{3}w + (\frac{2}{3}w)^2 + \cdots$ while for $|w| < 2$ we have $\frac{1}{1-\frac{w}{2}} = 1 + \frac{w}{2} + (\frac{w}{2})^2 + \cdots$. Therefore

$$\frac{5-3w}{(w-\frac{3}{2})(w-2)} = \sum_{n=0}^{\infty}(\frac{2}{3}(\frac{2}{3}w)^n + (\frac{w}{2})^n) = \sum_{n=0}^{\infty}((\frac{2}{3})^{n+1} + (\frac{1}{2})^n)w^n,\ \forall w;\ |w| < \frac{3}{2}.$$ Hence

$$\frac{4-6z}{2z^2-3z+1} = \sum_{n=0}^{\infty}((\frac{2}{3})^{n+1} + (\frac{1}{2})^n)(z+1)^n;\ |z+1| < \frac{3}{2}.$$

Comment This final series has centre -1 and radius of convergence of $\frac{3}{2}$. This approach via partial fractions and geometric series is much quicker than directly deriving the series by differentiation.

10. Cross-multiplication gives:

$$z = (e^z - 1)(1 + B_1 z + \frac{B_2}{2!}z^2 + \frac{B_3}{3!}z^3 + \cdots)$$

$$\Rightarrow z = (z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots)(1 + B_1 z + \frac{B_2}{2!}z^2 + \cdots)$$

$$\Rightarrow z = z + B_1 z^2 + \frac{B_2}{2!}z^2 + \frac{B_3}{3!}z^3 + \cdots$$
$$+\frac{1}{2!}(z^2 + B_1 z^3 + \frac{B_2}{2!}z^4 + \frac{B_3}{3!}z^5 + \cdots)$$
$$+\frac{1}{3!}(z^3 + B_1 z^4 + \frac{B_2}{2!}z^5 + \cdots) + \cdots$$
$$= z+z^2(B_1+\frac{1}{2})+z^3(\frac{B_2}{2}+\frac{B_1}{2}+\frac{1}{6})+z^4(\frac{B_3}{6}+\frac{B_2}{4}+\frac{B_1}{6}+\frac{1}{24})+z^5(\frac{B_4}{24}+\frac{B_3}{12}+\frac{B_2}{12}+\frac{B_1}{24}+\frac{1}{120})\cdots$$

Hence equating coefficients now gives $B_1 + \frac{1}{2} = 0$ so that $B_1 = -\frac{1}{2}$. Next $\frac{B_2}{2} = \frac{1}{4} - \frac{1}{6}$ so that $B_2 = \frac{1}{6}$; $B_3 = 6(-\frac{1}{24} + \frac{1}{12} - \frac{1}{24}) = 0$; $B_4 = 24(0 - \frac{1}{72} + \frac{1}{48} - \frac{1}{120}) = -\frac{1}{30}$.

Problem Set 6

1. Writng u_n to denote the nth term of the series we get;

$$|\frac{u_{n+1}}{u_n}| = |\frac{(1+i)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(1+i)^n}| = |\frac{1+i}{2}| = \frac{\sqrt{1^2+1^2}}{2} = \frac{\sqrt{2}}{2} < 1,$$

and so the series converges (absolutely).

2. Putting $z+1=0$ so we get that the centre is $z=-1$. Again take the absolute value of the ratio of successive terms gives and boundi it strictly by 1:

$$|\frac{3^{2(n+1)}(1+z)^{3(n+1)}}{3^{2n}(1+z)^{3n}}| = |3(1+z)^3| < 1$$

$$\Rightarrow |(1+z)|^3 < \frac{1}{3} \Rightarrow |1+z| < \frac{1}{\sqrt[3]{3}}.$$

Hence the radius of convergence is $\frac{1}{\sqrt[3]{3}}$.

3. Centre is $z=0$. As for the radius of convergence we put

$$|\frac{(n+2)5^{n+1}z^{n+2}}{(n+1)5^{n+2}z^{n+1}}| = |\frac{z(n+2)}{5(n+1)}| \to \frac{z}{5} \,\text{as}\, n \to \infty.$$

Hence the series converges if $\frac{|z|}{5} < 1$, so the radius of convergence is 5.

4.
$$|\frac{z_{n+1}}{z_n}| = |\frac{(n+1)!e^n(1+i)^n}{n!e^{n+1}(1+i)^{n+1}}| = |\frac{(n+1)}{e(1+i)}| \to \infty,$$

so the series diverges.

5.
$$e^{\text{Log}(z)} = e^{\ln|z| + \text{i}\text{Arg}(z)} = e^{\ln|z|}e^{i\text{Arg}(z)}$$
$$= |z|e^{i\text{Arg}(z)} = z.$$

6.
$$\text{Log}(i) = \ln|i| + i\text{Arg}(i) = 0 + i\tfrac{\pi}{2} = \tfrac{i\pi}{2}$$
$$\text{Log}(-i) = \ln|-i| + i\text{Arg}(-i) = 0 + i(-\frac{\pi}{2}) = -\frac{i\pi}{2}$$
$$\text{Log}(1+i) = \ln|1+i| + i\text{Arg}(1+i) = \ln(\sqrt{2}) + i\tfrac{\pi}{4} = \tfrac{1}{2}\ln 2 + i\tfrac{\pi}{4}$$
$$\text{Log}(4i) = \ln|4i| + i\text{Arg}(4i) = 2\ln 2 + i\tfrac{\pi}{2}.$$

7. Clearly both sequences approach the limit $\alpha e^{i\pi} = -\alpha < 0$. Now
$$\text{Log}(a_n) = \text{Log}(\alpha e^{i(\pi - \frac{1}{n})}) = \ln|\alpha| + i(\pi - \tfrac{1}{n}) \to \ln\alpha + i\pi;$$
however
$$\text{Log}(b_n) = \text{Log}(\alpha e^{i(\pi + \frac{1}{n})}) = \ln|\alpha| + i(-\pi + \tfrac{1}{n}) \to \ln\alpha - i\pi.$$

8. Here we have $u(r,\theta) = \ln(r)$ and $v(r,\theta) = \theta$
$$u_r(r,\theta) = \frac{1}{r},\ u_\theta(r,\theta) = 0, \quad \text{and}$$
$$v_r(r,\theta) = 0,\ v_\theta(r,\theta) = 1;$$
and so recalling the formulae from Question 10 Set 3, we see that the CR equations in polar form, which are:
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$
are satisfied, remembering that $v(r,\theta)$ here is only differentiable for $r \neq 0$ and $-\pi < \theta < \pi$ (so there is no discontinuity in $v(r,\theta)$ in the domain of definition).

9.
$$\begin{aligned}\log(z_1 z_2) &= \ln|z_1 z_2| + i\arg(z_1 z_2)\\ &= \ln|z_1| + \ln|z_2| + i(\arg z_1 + \arg(z_2))\\ &= (\ln|z_1| + i\arg(z_1)) + (\ln|z_2| + i\arg(z_2))\\ &= \log(z_1) + \log(z_2).\end{aligned}$$
Similarly
$$\begin{aligned}\log\big(\frac{z_1}{z_2}\big) &= \ln\big|\frac{z_1}{z_2}\big| + i\arg\big(\frac{z_1}{z_2}\big)\\ &= \ln\big(\frac{|z_1|}{|z_2|}\big) + i(\arg(z_1) - \arg(z_2))\\ &= (\ln|z_1| + i\arg(z_1)) - (\ln(|z_2| + i\arg(z_2))\\ &= \log(z_1) - \log(z_2).\end{aligned}$$

10. Here we have $z_1 = -2i$ and $z_2 = -i$:
$$\log(z_1) = \log(-2i) = \ln 2 + i(-\frac{\pi}{2} + 2n\pi),\ n \in \mathbb{Z},$$

$$\log(z_2) = \log(-i) = i(-\frac{\pi}{2} + 2n\pi),\ n \in \mathbb{Z};$$

noting that $z_1 z_2 = (-2i)(-i) = 2i^2 = -2$, we compare this to

$$\log(z_1 z_2) = \log(-2) = \ln 2 + i(\pi + 2n\pi),\ n \in \mathbb{Z}$$

and so we we see that $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ as the arg term in both cases runs over all odd integer multiples of π. However

$$\mathrm{Log}(z_1) = \ln 2 - i\tfrac{\pi}{2},\ \mathrm{Log}(z_2) = -i\tfrac{\pi}{2},$$

$$\mathrm{Log}(z_1 z_2) = \mathrm{Log}(-2) = \ln 2 + i\pi,$$

and so

$$\mathrm{Log}(z_1) + \mathrm{Log}(z_2) = \ln 2 - i\pi \neq \mathrm{Log}(z_1 z_2).$$

Problem Set 7

1.
$$\frac{1}{z - z^2} = \frac{1}{z(1-z)} = \frac{1}{z}\big(1 + z + z^2 + \cdots\big)$$

$$= \frac{1}{z} + 1 + z + \cdots + z^n + \cdots = \sum_{n=-1}^{\infty} z^n,\ \ |z| < 1.$$

2.
$$\frac{3-z}{z^2 - z^4} = \frac{3-z}{z^2(1-z^2)} = \frac{3-z}{z^2}\big(1 + z^2 + z^4 + \cdots + z^{2n} + \cdots\big),\ \text{with } |z^2| < 1 \Leftrightarrow |z| < 1;$$

$= \dfrac{3}{z^2} - \dfrac{1}{z} + 3 - z + 3z^2 - z^3 + 3z^4 - z^5 + 3x^6 - z^7 + \cdots$, so centre is 0, radius of convergence 1.

3. Put $w = z - 1$ so that $z = w + 1$. Then

$$\frac{1}{1 - z^2} = \frac{1}{(1-z)(1+z)} = -\frac{1}{w(w+2)}$$

$$g(w) = \frac{-\frac{1}{2}}{w(1 + \frac{w}{2})} = -\frac{1}{2w}\big(1 - \frac{w}{2} + (\frac{w}{2})^2 - (\frac{w}{3})^2 + \cdots\big)$$

$$= -\frac{1}{2w} + \frac{1}{4} - \frac{w}{8} + \frac{w^2}{16} - \cdots,\ \text{provided } |\tfrac{w}{2}| < 1 \Leftrightarrow |w| < 2.;$$

$$\Rightarrow f(z) = -\frac{1}{2(z-1)} + \frac{1}{4} - \frac{z-1}{8} + \frac{(z-1^2}{16} - \cdots + (-1)^n \frac{(z-1)^n}{2^{n+2}} + \cdots,\ |z-1| < 2.$$

4 -7. By partial fractions $f(z) = \frac{1}{5}\big(\frac{1}{z+1} - \frac{1}{z^2+4}\big)$ and $f(z)$ is not defined at $z = -1$ and $z = \pm 2i$. This gives three regions to consider defined by the

inequalities (i) $|z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$. Overall $\frac{1}{1+z} = \sum_{n=0}^{\infty}(-1)^n z^n$, when $|z| < 1$ and

$$\frac{1}{4+z^2} = \frac{\frac{1}{4}}{1+\frac{z^2}{4}} = \frac{1}{4}\big(1 - \frac{z^2}{4} + \frac{z^4}{16} - \frac{z^6}{64} + \cdots + (-1)^n\frac{z^{2n}}{4} + \cdots\big), \text{ for } |z| < 2.$$

(i) For $|z| < 1$ both of these series converge, so

$$f(z) == \frac{1}{5}\big(\frac{1}{z+1} - \frac{1}{z^2+4}\big)$$

$$= \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n z^n - \frac{1}{20}\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{4^n}$$

$$= (\frac{1}{5} - \frac{1}{20}) - \frac{1}{5}z + (\frac{1}{5} + \frac{1}{80})z^2 - \frac{1}{5}z^3 + (\frac{1}{5} - \frac{1}{320})z^4 + \cdots.$$

(ii) For 1<|z|<2 , the series for $\frac{1}{4+z^2}$is valid but not the one for $\frac{1}{z+1}$. Instead we use

$$\frac{1}{z+1} = \frac{1}{z(1+\frac{1}{z})} = \frac{1}{z}\big(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots), \text{ provided} |\tfrac{1}{z}| < 1 \Leftrightarrow |z| :> 1.$$

Hence in region (ii) we have a proper Laurent series:

$$f(z) = \frac{1}{5z}\sum_{n=0}^{\infty}(-1)^n\frac{1}{z})^n - \frac{1}{20}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n}.$$

While for region (iii) the Laurent series for $\frac{1}{z+1}$ is valid but we will need the corresponding series for $\frac{1}{4+z^2}$.

$$\frac{1}{z^2(1+\frac{4}{z^2})} = \frac{1}{z^2}\sum_{n=0}^{\infty}(-1)^n\big(\frac{2}{z}\big)^n.$$

Summing the two series gives for this region:

$$f(z) = \frac{1}{5z} - \frac{2}{5z^2} + \frac{1}{5z^3} + \frac{3}{5z^4} + \frac{1}{5z^5} - \frac{17}{5z^6} + \cdots.$$

8. $f(z) = \frac{\cos 2z}{z^4}$ is singular at z_0 but is analytic otherwise. The pole at $z_0 = 0$ is of order 4 so

$$\text{Res}_{z=0}f(z) = \tfrac{1}{(4-1)!}\tfrac{d^{4-1}}{dz^{4-1}}\big((z-0)^4\tfrac{\cos 2z}{z^4}\big)|_{z=0}$$

$$= \frac{1}{3!}\frac{d^3}{dz^3}(\cos 2z)|_{z=0} = \frac{1}{6}\cdot 2^3 \sin 2z|_{z=0} = 0.$$

9. Here $f(z) = \frac{z^2+11z+1}{(z+1)^2(z-2)}$ has a pole of order 2 and $z = -1$ and a simple pole at $z = 2$. For the latter case put $P(z) = z^2+11z+1$ and $Q(z) = (z+1)^2(z-2) = z^3 - 3z - 2$ to find

$$\text{Res}_{z=2}f(z) = \tfrac{P(2)}{Q'(2)} = \tfrac{z^2+11z+1}{3z^2-3}\big|_{z=2} = \tfrac{27}{9} = 3.$$

Using the formula for a pole of any order we get

$$\text{Res}_{z=-1} f(z) = \tfrac{1}{(2-1)!} \cdot \tfrac{d^{2-1}}{dz^{2-1}} \big((z+1)^2 \cdot \tfrac{z^2+11z+1}{(z+1)^2(z-2)} \big) \big|_{z=-1}$$

$$= \frac{d}{dz} (\frac{z^2+11z+1}{z-2}) \big|_{z=-1} = (\frac{z^2-4z-23}{(z-2)^2}) \big|_{z=-1} = \frac{-18}{9} = -2.$$

10. $f(z) = \sec z$ is singular when $\cos z = 0$, which is when $z = n\pi + \frac{\pi}{2}$ $\forall n \in \mathbb{Z}$. Now $Q(z) = \cos z$ has $Q'(z) = -\sin z$, which is not zero at any of these singular points so the poles are simple. Putting $P(z) = 1$ and $Q(z) = \cos z$ then gives

$$\text{Res}_{z=n\pi+\frac{\pi}{2}} f(z) = \tfrac{P(z)}{Q'(z)} \big|_{z=n\pi+\frac{\pi}{2}} = \tfrac{1}{-\sin z} \big|_{z=n\pi+\frac{\pi}{2}} = \pm 1$$

according as $z = 2n\pi + \frac{\pi}{2}$ or $z = (2n+1)\pi + \frac{\pi}{2}$.

Problem Set 8

1.
$$z^3 - 2iz^2 - z = 0 \Leftrightarrow z(z^2 - 2iz - 1) = 0 \Leftrightarrow z(z-i)^2 = 0$$
$$\Leftrightarrow z = 0 \text{ or } z = i.$$

Hence $f(z)$ has zeros at $z = 0$ and $z = i$; $f'(z) = 3z^2 - 4iz - 1$, $f''(z) = 6z - 4i$. Then $f'(0) = -1 \neq 0$ so that $f(z)$ has a simple zero at $z_0 = 0$. And $f'(i) = -3 + 4 - 1 = 0$ but $f''(i) = 6i + 4 \neq 0$ so that $f(z)$ has a zero of order 2 at i.

2. $\tan z = 0 \Leftrightarrow \sin z = 0 \Leftrightarrow z = n\pi$ $(n \in \mathbb{Z})$. Next, $f'(z) = \sec^2 z$ so $f'(n\pi) = \frac{1}{\pm 1} \neq 0$ so that all the zeros $n\pi$ of $f(z)$ are simple zeros.

3.
$$f(z) = \frac{z}{4z^2-1} = \frac{1}{4} \cdot \frac{z}{z^2 - \frac{1}{4}}$$

so that $f(z)$ has a *simple pole* at each of $z = \pm\frac{1}{2}$, each of which lie inside of C. Putting $P(z) = z$ and $Q(z) = z^2 - \frac{1}{4}$ so that $Q'(z) = 2z$. Hence

$$\text{Res}_{z=\frac{1}{2}} f(z) = \tfrac{z}{2z}\big|_{z=\frac{1}{2}} = \tfrac{1}{2} = \text{Res}_{z=-\frac{1}{2}} f(z).$$

Hence
$$\int_C \frac{z\,dz}{4z^2-1} = \frac{2\pi i}{4}(\frac{1}{2} + \frac{1}{2}) = \frac{\pi i}{2}.$$

4. $\cos z = 0 \Leftrightarrow z = n\pi + \frac{\pi}{2}$, $(n \in \mathbb{Z})$. None of these zeros lie inside of C, so by Cauchy's theorem $\int_C \frac{e^z\,dz}{\cos z} = 0$.

5. The integrand $\sin(z^2)$ is analytic inside C so that $\int_{\Gamma_R} \sin(z^2)\,dz = 0$.

6. The integrand $f(z)$ is singular only at $z = 1$, which lies inside Γ_R and $(z-1)f(z)$ is not singular at $z = 1$, which is therefore a simple pole. The required residue is

$$b_1 = \lim_{z\to 1} -\frac{e^{z^2}}{1} = -e$$

$$\Rightarrow \int_{\Gamma_R} \frac{e^{z^2}}{1-z}\, dz = -e.$$

7. Here $f(z)$ has singularities at ± 1, and both these points lie within the contour Γ_R. Now

$$\lim_{z \to 1} \frac{\sin(z-1)}{(z-1)(z+1)} = \lim_{z \to 1} \frac{\sin(z-1)}{z-1} \cdot \lim_{z \to 1} \frac{1}{z+1} = 1 \cdot \frac{1}{2} = \frac{1}{2},$$

so that $z = 1$ represents a *removable singularity* of $f(z)$. On the other hand,

$$\lim_{z \to -1} (z+1) \cdot \frac{\sin(z-1)}{(z+1)(z-1)} = \frac{\sin(-2)}{-2} = \frac{1}{2} \sin 2 \text{ and so}$$

$$\int_{\Gamma_R} \frac{\sin(z-1)}{z^2-1}\, dz = \frac{1}{2} \sin 2.$$

8. By the ML-inequality we have that

$$\int_{\Gamma} |F(z)|\, dz \le \frac{M}{R^k} \cdot \pi R = \frac{\pi M}{R^{k-1}}$$

since the length of the arc Γ is πR. Then

$$\lim_{R \to \infty} \int_{\Gamma} |F(z)|\, dz = 0 \Rightarrow \lim_{R \to \infty} \int_{\Gamma} F(z)\, dz = 0.$$

9. For $z = Re^{i\theta}$,

$$|F(z)| = |\frac{1}{R^6 e^{i6\theta} + 1}| \le \frac{1}{|R^6 e^{6i\theta}| - 1} = \frac{1}{R^6 - 1} \le \frac{2}{R^6}$$

if R is sufficiently large ($R > 2$ suffices), so we take $M = 2$, $k = 6$.

Comment Note we used the inequality $|z_1 + z_2| \ge |z_1| - |z_2|$ with $z_1 = R^6 e^{6i\theta}$ and $z_2 = 1$.

10. Let C be the closed contour consisting of the line from $-R$ to R followed by traversing the semicircle Γ centred at 0 of radius R above the real axis. The roots of $z^6 + 1$ are $z = e^{ik\pi/6}$, where $k = 1, 3, 5, 7, 11$ and these all represent simple poles of $z^6 + 1$ and the only ones inside of C are when $k = 1, 3, 5$. The respective residues are, (using L'Hopital's Rule)

$$\lim_{z \to \frac{\pi}{6}} \big((z - e^{\frac{i\pi}{6}})\frac{1}{z^6+1}\big) = \lim_{z \to e6^{\frac{\pi}{6}}} \frac{1}{6z^5} = \frac{1}{6} e^{-\frac{5\pi}{6}};$$

$$\lim_{z \to \frac{3\pi}{6}} \big((z - e^{\frac{i3\pi}{6}})\frac{1}{z^6+1}\big) = \lim_{z \to e6^{\frac{3\pi}{6}}} \frac{1}{6z^5} = \frac{1}{6} e^{-\frac{5\pi}{2}};$$

$$\lim_{z \to \frac{5\pi}{6}} \big((z - e^{\frac{i5\pi}{6}})\frac{1}{z^6+1}\big) = \lim_{z \to e6^{\frac{5\pi}{6}}} \frac{1}{6z^5} = \frac{1}{6} e^{-\frac{25\pi}{6}};$$

$$\Rightarrow \oint_C \frac{dz}{z^6+1} = 2\pi i\Big(\frac{1}{6}(e^{-\frac{5\pi i}{6}} + e^{-\frac{5\pi i}{2}} + e^{-\frac{25\pi i}{6}})\Big) = \frac{2\pi}{3}.$$

$$\therefore \int_{-R}^{R} \frac{dx}{x^6+1} + \int_\Gamma \frac{dz}{z^6+1} = \frac{2\pi}{3}.$$

Taking the limit as $R \to \infty$ and using the result of Question 9 then gives:

$$\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}.$$

Problem Set 9

1.

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})(u+iv)$$
$$= \frac{1}{2}((u_x - v_y) + i(u_y + v_x))$$

but by the Cauchy-Riemann equation, $u_x = v_y$ and $u_y = -v_x$ so that $\frac{\partial f}{\partial \bar{z}} = 0$.

2.

$$\frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})(x+iy) = \frac{1}{2}(1+0-0+1) = 1;$$
$$\frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})(x-iy) = \frac{1}{2}(1+0+0+1) = 1.$$

3. The points a and b both belong to the interior of γ, so we can apply Cauchy's theorem for multi-connected domains to obtain:

$$I = \int_\gamma \frac{dz}{(z-a)^n(z-b)^n} = \int_{\gamma_a} \frac{(z-b)^{-n}}{(z-a)^n}\,dz + \int_{\gamma_b} \frac{(z-a)^{-n}}{(z-b)^n}\,dz = I_1 + I_2$$

where γ_a and γ_b are contours around the points a and b respectively.

Now by the Cauchy integral formula we get

$$I_1 = \frac{2\pi i}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}((z-b)^{-n})|_{z=a} = \frac{2\pi i}{(n-1)!}(-1)^{n-1}\frac{(2n-2)!}{(n-1)!}\cdot\frac{1}{(a-b)^{2n-1}}$$

and in exactly the same way we get

$$I_2 = \frac{2\pi i}{(n-1)!}(-1)^{n-1}\frac{(2n-2)!}{(n-1)!}\cdot\frac{1}{(b-a)^{2n-1}};$$

Therefore $I = I_1 + I_2 = 0$.

4. In this case b lies outside of γ so that $I = \int_{\gamma_a}$ and so by Question 4:

$$I = \frac{2\pi i}{(n-1)!}(-1)^{n-1}\frac{(2n-2)!}{(n-1)!} \cdot \frac{1}{(a-b)^{2n-1}}.$$

5. The integrand is now analytic inside and on the boundary of γ so by the CIF we have that $I = 0$.

6. The integrand has a pole of order 4 at the point $z = -2$ interior to the circle $|z+1| = 2$ so the CIF gives

$$I = \frac{2\pi i}{3!}\frac{d^3}{dz^3}(\sin z)|_{z=-2} = -\frac{\pi i}{3}\cos 2.$$

7. Use the McLaurin series for cosine:

$$\cos z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$$\Rightarrow \cos\sqrt{z} = \sum_{n=0}^{\infty}(-1)^n \frac{z^n}{(2n)!} = 1 - \frac{z}{2!} + \frac{z^2}{4!} - \cdots.$$

Comment The potential ambiguity in the $\sqrt{}$ sign is inconsequential here as cosine is an even function.

8. Now

$$4x^2 + y^2 - 2y = 0 \Leftrightarrow 4x^2 + (y-1)^2 = 1$$

$$\Leftrightarrow \frac{x^2}{(1/2)^2} + (y-1)^2 = 1;$$

so γ is an ellipse centred at $(0,1)$ with x-axis $[-\frac{1}{2}, \frac{1}{2}]$ and y-axis $[0,2]$. The denominator of the integrand is $(z+i)^2(z-i)^2$. The singularities are at $\pm i$ of which only i lies within γ. Hence apply the CIF with $f(z) = \frac{e^{\pi z}}{(z+i)^2}$ and $z_0 = i$. Hence

$$I = 2\pi i f'(z_0) = 2\pi i[\frac{e^{\pi z}}{(z+i)^2}]'|_{z=i}$$

$$= 2\pi i\{\frac{\pi e^{\pi z}(z+i)^2 - 2e^{\pi z}(z+i)}{(z+i)^4}]_{z=i}$$

$$= 2\pi i[e^{\pi z}(\frac{\pi z + \pi i - 2}{(z+i)^3})]_{z=i}$$

$$= 2\pi i e^{\pi z}(\frac{2\pi i - 2}{-8i}) = \frac{\pi}{2}e^{\pi i}(1 - \pi i).$$

9.

$$\frac{|p(z)|}{|z|^n} \geq |a_n| - (\frac{|a_{n-1}|}{|z|} + \cdots + \frac{|a_0|}{|z|^n})$$

which implies that for all z with $|z|$ sufficiently large

$$\Rightarrow \frac{|p(z)|}{|z|^n} \geq \frac{|a_n|}{2} > 0;$$

in particular there exists a bound K such that for all z such that $|z| > K$, $|p(z)| > 1$ so that $|f(z)| < 1$ for all z outside the circle of radius K centred at the origin. Since $f(z)$ is continuous there exists a bound $M > 1$ such that $|f(z)| \leq M$ for all z such that $|z| \leq K$. Therefore $|f(z)| \leq M$ for all $z \in \mathbb{C}$, which is to say that $f(z)$ is a bounded function.

10. We complete the proof of the Fundamental theorem of algebra by showing that any bounded differentiable function $f(z)$ is constant. This together with Question 8 yields a contradiction to the assumption that $p(z)$ has no root. We do this by showing that $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$ by the Cauchy integration formula for the first derivative. Since $f(z)$ is differentiable everywhere we have

$$f'(z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)\,dz}{(z-z_0)^2}.$$

By Question 8 we have that $|f(z)| \leq M$ for some bound M. However, since $|z-z_0|^2 = r^2$ on this contour and so we have by the ML-inequality that

$$|f'(z_0)| \leq \frac{1}{2\pi}\int_{|z-z_0|=r} |\frac{f(z)}{(z-z_0)^2}|\,dz \leq \frac{1}{2\pi}\cdot\frac{M}{r^2}\cdot 2\pi r = \frac{M}{r}.$$

Since r is arbitary we see by letting $r \to \infty$ that $|f'(z_0)| \leq 0$ and so $f'(z_0) = 0$, which, as we have already noted, completes the proof.

Problem Set 10

1. $n^s = e^{(\ln n)s}$ so that

$$n^s = \sum_{k=0}^{\infty} \frac{(\ln(n)s)^k}{k!}$$

$$\Rightarrow \zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \sum_{n=1}^{\infty}(\sum_{k=0}^{\infty}\frac{(\ln(n)s)^k}{k!})^{-1}.$$

2.

$$(1-\frac{1}{p^s})^{-1} = \frac{1}{1-\frac{1}{p^s}} = \sum_{k=0}^{\infty}\frac{1}{p^{sk}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots. \qquad (4)$$

3. Consider the expansion of $\Pi_p(1-\frac{1}{p^s})^{-1}$ that comes from replacing each term in the product by the corresponding series as in (4). We get the sum of the reciprocals of all prime products, each raised to the power s. By the Fundamental theorem of arithmetic, each positive integer n^s occurs once and (by uniqueness) once only as a denominator term so we conclude:

$$\Pi_p(1-\frac{1}{p^s})^{-1} = \sum_{n=1}^{\infty}\frac{1}{n^s} = \zeta(s).$$

4.

$$(1-2^{1-s})\zeta(s) = (1-2\cdot\frac{1}{2^s})(1+\frac{1}{2^s}+\frac{1}{3^s}+\cdots)$$

$$= (1+\frac{1}{2^s}+\frac{1}{3^s}+\cdots) - 2(\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{6^s}+\cdots)$$

$$= 1-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+\frac{1}{5^s}-\frac{1}{6^s}+\cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^s}$$

$$\therefore \zeta(s) = \frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^s}$$

and since the series on the right converges for all s such that $\mathbb{R}(s) > 0$, we see that this formula defines a function that extends $\zeta(s)$ to all values of s to the right of the imaginary axis (except 1).

5.

$$\frac{z^2}{n^2\pi^2 - z^2} = \frac{z^2}{n^2\pi^2}\cdot\frac{1}{1-\frac{z^2}{n^2\pi^2}} = \frac{z^2}{n^2\pi^2}\sum_{k=0}^{\infty}(\frac{z^2}{n^2\pi^2})^k.$$

Hence from the Euler identity:

$$z\cot z = 1 - 2\sum_{n=1}^{\infty}\frac{z^2}{n^2\pi^2}(\sum_{k=0}^{\infty}(\frac{z}{n\pi})^{2k}) = 1 - 2\sum_{k=0}^{\infty}(\sum_{n=1}^{\infty}\frac{1}{n^{2k+2}})\frac{z^{2k+2}}{\pi^{2k+2}})$$

$$= 1 - 2\sum_{k=1}^{\infty}(\sum_{n=1}^{\infty}\frac{1}{n^{2k}})\frac{z^{2k}}{\pi^{2k}}.$$

6.

$$z\cot z = z\frac{\cos z}{\sin z} = z\frac{\frac{1}{2}(e^{iz}+e^{-iz})}{\frac{1}{2i}(e^{iz}-e^{-iz})} = iz\frac{e^{2iz}+1}{e^{2iz}-1} = \frac{2iz}{e^{2iz}-1} + \frac{ize^{2iz}-iz}{e^{2iz}-1}$$

$$= \frac{iz(e^{2iz}-1)}{e^{2iz}-1} + \frac{2iz}{e^{2iz}-1} = iz + \frac{2iz}{e^{2iz}-1}.$$

7. By Question 10 Set 5 we have $\frac{z}{e^z-1} = \sum_{k=0}^{\infty}\frac{B_k z^k}{k!}$, so applying this to the result of the previous problem we obtain:

$$z\cot z = iz + \sum_{k=0}^{\infty}\frac{B_k(2iz)^k}{k!}$$

and since $B_0 = 1$ and $B_1 = -\frac{1}{2}$ we have by Question 5:

$$z\cot z = 1 + \sum_{k=2}^{\infty}\frac{B_k(2iz)^k}{k!} = 1 - 2\sum_{k=1}^{\infty}\zeta(2k)\frac{z^{2k}}{\pi^{2k}}.$$

8. Equating the coefficients of z^{2k} we get that for all integers $k \geq 1$:

$$\frac{B_{2k}2^{2k}i^{2k}}{(2k)!} = -\frac{2\zeta(2k)}{\pi^{2k}}$$

$$\Rightarrow \zeta(2k) = \frac{(-1)^{k+1}2^{2k}\pi^{2k}B_{2k}}{2(2k)!} \tag{5}$$

9.

$$\sum_{j=0}^{k} \binom{k+1}{j} B_j = 0,\ B_0 = 1$$

and so $B_0 + \binom{2}{1}B_1 = 0$ whence $B_1 = -\frac{1}{2}$. (However $B_{2k+1} = 0$ for all $k \geq 1$.) Next

$$B_0 + \binom{3}{1}B_1 + \binom{3}{2}B_2 = 0$$

$$\Rightarrow 1 - \frac{3}{2} + 3B_2 = 0 \Rightarrow B_2 = \frac{1}{6};$$

$$B_0 + \binom{4}{1}B_1 + \binom{4}{2}B_2 + \binom{4}{3}B_3 = 0$$

$$\Rightarrow 1 - 2 + 1 + B_3 = 0 \Rightarrow B_3 = 0;$$

$$B_0 + \binom{5}{1}B_1 + \binom{5}{2}B_2 + \binom{5}{4}B_4 = 0$$

$$\Rightarrow 1 - \frac{5}{2} + \frac{10}{6} + 5B_4 = 0$$

$$\Rightarrow B_4 = \frac{-6 + 15 - 10}{30} = -\frac{1}{30}.$$

$$\Rightarrow \zeta(2k) = \frac{(-1)^{k+1}2^{2k}\pi^{2k}B_{2k}}{2(2k)!}$$

From (9) we have for $k = 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{(-1)^2 2^2 \pi^2 \cdot \frac{1}{6}}{2(2!)} = \frac{4\pi^2}{24} = \frac{\pi^2}{6};$$

and for $k = 2$:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \zeta(4) = \frac{(-1)^3 2^4 \pi^4 B_4}{2(4!)} = \frac{16\pi^4}{2(24)(30)} = \frac{\pi^4}{90}.$$

10.

$$\zeta(s) = 2(2\pi)^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s),\ \mathbb{R}(s) < 1$$

Putting $s = -2k$ in the functional equation we see that the factor $\sin(\frac{\pi s}{2}) = \sin(-k\pi) = 0$ so that $\zeta(-2k) = 0$.

Comment The negative even integers are known as the *trivial* zeros of the zeta function. The more interesting ones are the ones in the *critical strip* $0 < \mathbb{R}(z) < 1$: the celebrated *Riemann hypothesis* is that all these zeros have $\mathbb{R}(z) = \frac{1}{2}$.

On the other hand taking $s = -2k + 1$ we obtain:

$$\zeta(-2k+1) = 2(2\pi)^{-2k-2}\sin(\frac{\pi(-2k+1)}{2})\Gamma(2k)\zeta(2k)$$

$$= \frac{\Gamma(2k)(-1)^{k+1}2^{2k+1}\pi^{2k}B_{2k}}{2^{2k+2}\pi^{2k+2}} = \frac{(-1)^{k+1}\Gamma(2k)B_{2k}}{4\pi^2}.$$