Towards a better dartboard

The standard numbering sequence of a modern dartboard is often credited to Brian Gamblin, from the English town of Bury, who introduced it in 1896. Although this is disputed, (there seems to be no evidence the man even existed) the standard British dartboard sequence does date to the early years of the 20th century. The circular layout of the numbers 1 - 20 however seems to lack any natural symmetries or uniformities, and so today you are invited to look into this. But we will begin with some smaller rings of numbers.

A necklace has $2n$ ($n \geq 2$) beads coloured alternately black (B) and red (R). Can we label the beads with the full set of integers $1, 2, \cdots, 2n$ so that the labels of each RBR-trio of beads sums to the same integer? Today we look at some particular cases.

Problem 1 Show that the answer to the previous question is 'no' if $n = 2$ and is 'yes' if $n = 3$.

Problem 2 Show that any solution for $n = 3$ is also a solution for each BRB-trio as well.

Problem 3 Show that for $n \geq 4$ there is no labelling that is simultaneously an RBR-solution and a BRB-solution.

And finally, the dartboard case.

Problem 4 Find a solution for $n = 10$, which is to say write the integers $1-20$ in a circle with the numbers coloured alternately black and red, beginning with a black 20, in such a way that each consecutive RBR-sequence sums to the same number.

Towards a better dartboard solutions

Problem 1 $n = 2$. A constant RBR-sum labelling is impossible in this case as the two RBR-sequences share the same pair of red labels but have different blue labels. The same argument applies to BRB sums.

For $n = 3$ however there is the (circular) solution $1, \mathbf{5}, 3, \mathbf{4}, 2, \mathbf{6}$. The bold labels represent black. The RBR-sums are indeed all equal to 9:

$$1 + \mathbf{5} + 3 = 3 + \mathbf{4} + 2 = 2 + \mathbf{6} + 1 = 9.$$

Problem 2 We note in the previous example that the BRB trios also sum to a common value (of 12). Suppose we have an RBR-labelling $a_0, a_1, \cdots, a_5$ of the cycle C_6, with a_0 red, such that each RBR-sum is the same. We then have equations $a_0 + a_1 + a_2 = a_2 + a_3 + a_4 = a_4 + a_5 + a_0$, which imply

$$a_0 + a_1 = a_3 + a_4 \tag{1}$$

$$a_2 + a_3 = a_5 + a_0 \tag{2}$$

Reversing equation (2) and subtracting it from (1) then gives

$$(a_1 - a_5 = a_4 - a_2) \Leftrightarrow (a_1 + a_2 = a_4 + a_5) \Leftrightarrow (a_1 + a_2 + a_3 = a_3 + a_4 + a_5). \tag{3}$$

Similar reasoning gives $a_3 + a_4 + a_5 = a_5 + a_0 + a_1$, thereby showing that all three BRB-sums also are equal. By symmetry, equality of all BRB-sums likewise implies equality of RBR-sums when $n = 3$.

Problem 3 We suppose that the cycle C_{2n} ($n \geq 4$) is labelled in such a way that all RBR-sums are equal to one another, as are all BRB-sums. We denote the cyclic sequence of labels by $a_0, a_1, \cdots, a_{2n-1}$.

Taking subscripts modulo $2n$, for any subscript i we have the equations:

$$a_i + a_{i+1} = a_{i+3} + a_{i+4} \tag{4}$$

$$a_{i+1} + a_{i+2} = a_{i+4} + a_{i+5} \tag{5}$$

$$a_{i+2} + a_{i+3} = a_{i+5} + a_{i+6}. \tag{6}$$

From (5) we obtain $a_{i+2} - a_{i+5} = a_{i+4} - a_{i+1}$. We apply this to (6) and then employ (4) to deduce that:

$$a_{i+6} = a_{i+2} + a_{i+3} - a_{i+5} = a_{i+3} + a_{i+4} - a_{i+1} = a_i.$$

Therefore for any i we have $a_i = a_{i+6}$. However, since $2n \geq 8$, it follows that i and $i + 6$ are distinct modulo $2n$. From this contradiction it follows that no labelling may simultaneously have constant RBR- and BRB-sums.

Problem 4 We begin with 20, which is traditionally the topmost sector of the board, coloured black, and imagine the successive clockwise sectors numbered as follows:

20 4 **16** 7 **12** 8 **14** 5 **13** 9 **17** 1 **11** 15 **10** 2 **19** 6 **18** 3.

The total score of each RBR sector is then 27. For comparison, here is the standard dartboard numbering.

20 1 **18** 4 **13** 6 **10** 15 **2** 17 **3** 19 **7** 16 **8** 11 **14** 9 **12** 5.