Mathematics 108 Mechanics

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March 10, 2018

Mechanics was the birthplace of calculus, which is the mathematics governing motion. The topics covered here begin with simple kinematics and problems involving constant acceleration (the *SUVAT equations*) and, in Set 2, *projectile motion*. In Set 3 we move on to *dynamics*, the study of the causes of motion, which is centred on *Newton's Second Law* $F = ma$ followed by the notions of energy and work, which features problems involving *Hooke's Law* of linear springs (Sets 4 and 5). In Set 8 we meet the related notions of *impulse* and *power*.

To deal with *circular motion* and acceleration (Set 6) requires development of the equations of velocity and acceleration in terms of *radial* and *transverse* components, which refer to an orthogonal pair of unit vectors at the point P in question, which are not fixed but whose direction is determined by the radial vector from the origin to the point P. Since the direction of the local radial unit vector $\hat{\mathbf{r}}$ is a function of the polar angle θ, this interaction leads to more complicated formulae describing the motion of a particle but they do allow for analysis of problems that feature rotation, and in particular we visit problems involving pendulums of various types and planetary motion. In Set 7 we deal with simple examples involving celestial motion under the simplifying assumption that that motion is uniform and circular. However, in Set 10, we deal with the general case of elliptical and other conic orbits. Preparation for this is in Set 9 on the nature of conic sections, with a view to their application in orbital theory. The mathematics of the final set becomes quite sophisticated in that the student will apply several facets of the year's work including solution of second order differential equations and calculus involving polar form.

Problem Set 1 Motion

1. I can walk to work at 4mph or cycle at 12mph and I save ten minutes if I ride my bike. How far is it to work?

2. A cyclist rides along a straight road from A to B and back again, maintaining a constant speed, *relative to the air*, of 30km/hr. There is a 5km/hr breeze from A to B. Given that he rides for two hours in all, find the distance from A to B.

3. Five trains per hour travel from London to Colchester in each direction. How many will you meet going in the opposite direction given that your journey takes an hour? What if it takes two hours?

4. Two trains, each travelling at 25mph, are fifty miles apart and heading towards each other. A fly on the front of one of the trains, which flies at 50mph, flies towards the other train and, when it reaches it, instantly flies back to the other, and keeps repeating this until the trains collide. What is the total distance flown by the fly?

5. A 5 metre ladder, leaning against a wall, slips so that its base moves away from the wall at a rate of 2m/sec. How fast will the top of the ladder be moving down the wall when the base is 4 metres from the wall?

6. Einstein's Theory of Special Relativity is based on the fact that the mass of a moving object m with rest mass m_0 is given by:

$$m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}};$$

where v is the velocity of the object and c is the speed of light in a vacuum. In terms of c, at what velocity will the mass of the object be double its rest mass?

7. A particle moves along the x-axis in a straight line in such a way $x = A \sin nt + B \cos nt$ where t denotes time. Find the acceleration $\ddot{x}$ of the particle as a function of x.

8. For the particle in Question 8, find the range of its motion and, taking $A, B \geq 0$, find the first times when the particle reaches the extremities of its range interval.

9. There is a (cylindrical) glass, 4 inches high, circumference 6 inches. Inside the glass, 1 inch from the top, is a drop of honey. Outside of the glass, 1 inch from the bottom, on the opposite side, is an ant. What is the minimum distance that the ant must walk to get to her honey?

10. The hands of a clock point in the same direction at 12 o'clock. They also do this a little after 1:05, and a little more after 2:10, and so on. Exactly how much time elapses after midday until the hands of a clock face again point in the same direction?

Problem Set 2 Kinematics and the SUVAT Equations

For a particle P moving in a straight line with constant acceleration a we find by integrating with respect to time t that $v = u + at$, where u denotes the initial velocity of P and v denotes its velocity after time t. Integrating a second time we obtain $s = \frac{1}{2}at^2 + ut$, where s denotes the position of P with respect to its initial position.

1. Eliminate t from the previous pair of equations to express v^2 in terms of u^2, a, and s.

2. A train leaves a station with constant acceleration. After 20 seconds it has travelled 500m. What is its acceleration?

3. A particle moves from rest with a constant acceleration of 1m/sec^2. What is its speed when it has travelled 50m?

4. A cheetah is running at 6.20m/s and, with constant acceleration, speeds up to 23.1 m/s in a time of 3.3 s. How much ground does the animal cover during this time?

5. A ball is tossed up vertically at a speed of 7.7m/sec. Neglecting air friction, what is the maximum height reached by the ball, given that g, the *acceleration due to gravity* is 9.81m/sec^2?

6. How long is the ball in Question 5 in the air?

7. How high would the ball need to be thrown to be air borne for 6 seconds?

8. A projectile has x and y co-ordinates at time t given by

$$x = (V\cos\alpha)t, \; y = (V\sin\alpha)t - \frac{1}{2}gt^2$$

where V and α are the initial speed and angle of projection respectively. Express y in terms of x and the parameters, V, α, and $g \approx 10\,ms^{-1}$, the acceleration due to gravity.

9. Find the direction of the velocity of a projectile, which is projected at $30ms^{-1}$ at an angle of $45°$ to the horizontal, when it has travelled a horizontal distance of 30m.

10. A body is projected from the origin with speed $\sqrt{4g/h}$ and passes through the point with co-ordinates $(h, h/8)$. Find the two possible angles of projection.

Problem Set 3 Newton's Law

The statement that the acceleration a in the direction of a force F acting on a body of mass m is given by $F = ma$ is called *Newton's Second Law*. *Newton's First Law* is the special case where $F = 0$ so that $a = 0$ and so the velocity of a body in a particular direction does not change if the net force in that direction is zero. This is often called the *Principle of Inertia*. *Newton's Third law* is that to every action there is an equal and opposite reaction, meaning that if one object exerts a force on a second, then the second exerts the opposite force on the first. When we speak of Newton's Law, we shall mean the Second Law, which includes the first.

1. Find the force required to give a car of mass 2×10^3kg a uniform acceleration of $2 \cdot 5$m/sec^2.

2. Find the force required to accelerate the same car for 0 to 36km/hr in 10sec.

3. A person of mass $m$kg stands on some scales in a lift accelerating (upwards) at fm/sec^2. What do the scales read in Newtons?

4. A balloon of mass m ascends vertically with acceleration f. What is the upward thrust of the balloon due to its *buoyancy*? (which is the upward force acting on the balloon.)

5. If ballast of mass m_2 is thrown out of the balloon, what is the value of its new upward acceleration?

6. A stone is dropped from a balloon that is rising at 10m/sec and the stone hits the ground 8 seconds later. How high was the balloon when the stone was dropped?

7. A balloon of total mass M descends with a downward acceleration of f_1. Neglecting air resistance, find the mass of the ballast that needs to be discarded from the balloon in order for it to rise with an acceleration of f_2.

8. An aircraft of mass 10^5kg produces a thrust of $2 \cdot 5 \times 10^5$N, which then increases at a constant rate of 6×10^3N/sec . Find the time taken for it to achieve a speed of 180km/hr.

9. An object falls vertically past a window 2m tall in $\frac{1}{12}$sec. Find the height above the bottom of the window from which the object was dropped.

10. (Courtesy of MIT Opencourseware) While driving along the highway at 40 m/s, you spot another car 50m ahead, traveling at a constant speed of 30 m/s. You apply the brakes and begin decelerating at 1.0 m/sec^2. Will you catch the other car?

Problem Set 4 Hooke's law, energy, and work

We consider a spring of natural length l suspended from a fixed point. We assume that the force exerted by the spring obeys *Hooke's Law* in that $F = \frac{\lambda x}{l}$ where x is the extension (which may be negative) from the natural length l and $\lambda > 0$, is the *modulus of elasticity* of the spring.

1. A body of mass m is suspended from the lower end of the spring, which then extends to x_0 past its natural length. What is the value of λ for this spring?

2. A mass m oscillating vertically on the spring then has equation of motion about its equilibrium position described by:

$$m\ddot{x} = mg - \frac{\lambda}{l}x \tag{1}$$

Find the general solution of the differential equation (1), expressing it as a single cosine function involving two arbitrary constants.

3. Find the particular solution to (1) given that the stationary mass is stretched to a position a past its natural length and then released from rest.

4. Find the period, frequency, and the maximum displacement of m.

5. Express the velocity $v = \dot{x}$ in terms of the position x of the mass by writing $\ddot{x} = v\frac{dv}{dx}$ and integrating both sides with respect to x.

The *work* done by a (variable) force F when a body is moved in the direction of the force from a_1 to a_2 is given by $\int_{a_1}^{a_2} F\,dx$. For a constant force vector $\mathbf{F}$ the work is given by $\mathbf{F} \bullet \mathbf{x} = Fx\cos\theta$, where θ is the angle between the force and motion vector. In particular the work done by gravity in lifting a body of mass m through a height h is $W = mgh$ as the force F of gravity on the body is $F = ma = mg$.

6. Use the Chain rule to show that Write $\ddot{x} = \frac{d}{dx}(\frac{1}{2}\dot{x}^2)$.

7. Use Question 6 to find the work done by a possibly variable force F which causes a body of mass m to accelerate from speed v_1 to v_2 in travelling from $x = 0$ to $x = s$.

8. Use the previeous question to deduce that if a mass m falls from h_1 to h_2 with velocity increasing from v_1 to v_2 then

$$mgh_1 + \frac{1}{2}mv_1^2 = mgh_2 + \frac{1}{2}mv_2^2.$$

9. Find the work done by a force in stretching (or compressing) a spring with modulus of elasticity λ and natural length l to $l + a$.

10. A mass of 50kg is dropped from a height of $2 \cdot 2$m above the ground onto a vertical spring resting on the ground of natural length $0 \cdot 4$m. The closest approach of the mass to the ground is then $0 \cdot 2$m. Find the modulus of elasticity of the spring.

Problem Set 5 Conservation of Energy

1. A uniform thin chain of mass m and length 2 has half of its length lying over the edge of a smooth horizontal table with the remaining half extended at right angles to the edge on top. It is held in position and then released. Let $x(t)$ denote the distance below the table top of the lower end of the chain at time t after its release. Use Conservation of Energy to show that

$$\dot{x}(t) = \sqrt{\frac{g}{2}}\sqrt{x^2 - 1}.$$

2. Deduce the result of Question 2 by deriving the equation of motion.
3. Show that the solution $x(t)$ of Question 2 is given by

$$x(t) = \cosh\sqrt{\frac{g}{2}}t.$$

4. Compare the velocity of the sliding chain to that which the chain would have if it were suspended vertically with the top 1 metre above the height of the table and released.
5. Two objects of respective masses $m_1 < m_2$ are tied by a string placed over a smooth peg projecting horizontally from a wall. The masses are then released. Taking the upward direction as positive, write down equations of motion of each of the two masses in terms of their common acceleration a and the tension T in the string. Hence express a and T in terms of m_1, m_2 and g.
6. A bead slides on a smooth circular wire, radius a, which is fixed in a vertical plane. The bead is displaced slightly from the top point of the circle. Find the subsequent speed of the bead as a function of the angle θ formed by the radius of the circle through the position point P of the bead with the vertical.
7. A puck sits on top of a large hemisphere of ice of radius a. The puck is displaced so that it slides off. At what angle of the radius vector from the vertical does the puck leave the surface of the sphere?
8. A pendulum consists of a light inextensible rod of length l pivoted at one end with a mass m attached to the other. At the bottom of its swing, the pendulum has angular velocity ω. Show that the cosine of the angle the rod makes with the vertical at the top of the pendulum's swing is $1 - \frac{l\omega^2}{2g}$.
9. A pendulum consisting of a light rod of length l and bob mass m hangs suspended from inside the roof of a car, which has acceleration of a. Find the angle that the pendulum rod makes with the vertical.
10. An *elastic string* (which exerts a force when stretched but which goes slack under compression) of length l and modulus of elasticity λ is attached to a rigid beam. A particle of mass m is attached to the other end, held near the attached end of the string and let fall. Show that the particle falls a maximum distance x that satisfies the equation:

$$x^2 - (1 + \frac{mg}{\lambda})2lx + l^2.$$

Problem Set 6 Circular motion

Circular motion The vectors $\hat{\mathbf{r}} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ and $\hat{\theta} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$ represent an orthogonal pair of unit vectors in the plane with polar angle θ. These are known respectively as the *radial* and *transverse* components of a point P with polar co-ordinates (r, θ).

1. Express $\mathbf{i}$ and $\mathbf{j}$ in terms of $\hat{\mathbf{r}}$ and $\hat{\theta}$.

2. Express the time derivatives of $\hat{\mathbf{r}}$ and $\hat{\theta}$ in terms of those of r, θ and $\hat{\mathbf{r}}$ and $\hat{\theta}$.

3. By finding the time derivative of $r\hat{\mathbf{r}}$, find the velocity $\mathbf{v}$ in terms of the radial and angular velocity, $\dot{r}$ and $\dot{\theta}$ respectively.

4. Similarly find $\mathbf{a} = \dot{\mathbf{v}}$ as a linear combination of $\dot{\mathbf{r}}$ and $\dot{\theta}$.

5. A particle moves along the curve with equation $r = a(2 + \cos\theta)$ in the sense of increasing θ with constant speed v. Find $\dot{\theta}$ in terms of v and a.

6. Find the value of the *centripetal acceleration* towards the centre of circle for a particle rotating in a circle with constant angular velocity $\omega = \dot{\theta}$, expressing it both in terms of r and ω and in terms of r and v, the velocity of the particle.

Simple Pendulum

7. Consider a simple pendulum with bob P of mass m and length l and (variable) angle of θ from the vertical. Write down the expression for the acceleration $\mathbf{a}$ of P and the force $\mathbf{F}$ acting on P in terms of the tension T in the supporting string (light and inextensible).

8. Write down an second order differential equation for θ.

9. Replace $\sin\theta$ by θ in the equation of Question 9 (accurate for small θ) and solve. Hence find the approximate period of the simple pendulum.

10. *Conical pendulum* Let a pendulum of length l and bob mass m be spun in a horizontal circle with angular velocity ω (so that the string and bob trace out a cone with apex at the suspension point of the pendulum and whose axis passes through the centre of this circle of rotation). Determine the angle θ that the suspension string makes with the vertical in terms of ω and l.

Problem Set 7 Planetary motion

1. If it is noon at Greenwich what is the true time of day in a city in Siberia whose longitude is $45°$ east?

2. The moon is $\frac{1}{4}$ of a million miles away and the Sun is 93 million. Given that the moon just covers the Sun in a total eclipse (which it does), what is the ratio of the Moon's diameter to that of the Sun?

3. Following on from Question 2, how many moons could fit inside the sphere of the Sun?

4. To the nearest whole number, how many times per year does the Earth rotate on its axis?

5. *Kepler's Third Law* says that the square of the period (year) of a planet is proportional to the cube of the radius of its orbit. What is the length of the year of an asteroid that is twice as far from the Sun as is the Earth?

6. In 1543 Copernicus measured the length of the year of Saturn to be about $29 \cdot 5$ Earth years. How far is Saturn from the Sun? (Use the radius of Earth's orbit as one unit.)

7. Suppose the Earth describes a uniform circular motion around the Sun in one year ($T = 3 \cdot 16 \times 10^7$ s)where the radius $a = 1 \cdot 5 \times 10^{11}$m. Find the orbital speed of the Earth and its acceleration towards the Sun. Compare this acceleration to that due to Earth's gravity.

8. Given that the length of the day is $T = 8 \cdot 62 \times 10^4$s and consider a point P on the Earth's surface of latitude ϕ. Find the rotational velocity at P and the acceleration at P towards the Earth's rotational axis, given that the radius of the Earth is $r = 6 \cdot 4 \times 10^6$m. Find the maximum value of this acceleration and compare it to g.

9. Given that the mass M of the Earth ius 5.976×10^{24}kg and the Universal gravitational constant $\gamma = 6.670 \times 10^{-11} m^3 kg^{-1} s^{-1}$, find the radius of a synchronous satellite, which is to say the distance from the Earth's centre of a satellite that stay fixed with respect to the Earth's surface.

10. *Derivation of Kepler's 3rd Law for Circular orbit.* Let a planet P of mass m have a circular orbit of radius r around a star of mass M. By *Newton's law of gravitational attraction*, the radial force of attraction experienced by P is given by

$$F = \frac{\gamma M m}{r^2}$$

where γ is the *universal gravitational constant.* Show that the period (year) T of the planet is given by

$$T = \frac{2\pi}{\sqrt{\gamma M}} r^{\frac{3}{2}}.$$

Problem Set 8 Impulse and Power

1. A particle of unit mass strikes another of mass m and afterwards they rebound away from each other at equal speeds. What is the value of m?

2. The *impulse* I of a (possibly variable) force acting in the direction of motion of a mass m over a time interval from t_1 to t_2 is defined as:

$$I = \int_{t_1}^{t_2} F\,dt.$$

By using Newton's Law and a suitable change of variable show that $I = \Delta p$, the change in momentum of the mass.

3. If a certain mass has its velocity changed from 6.00 m/s to 7.50 m/s when a 3.00 N force acts for 4.00 seconds, find the mass of the moving object.

4. A golf ball of mass 100 g, initially at rest, is struck by a club. After the impact, the ball moves off with a velocity of 50.0 m/s. If the ball and club are in contact for $5 \cdot 00 \times 10^{-3}$seconds, what was the average force acting on the ball?

5. A 3.00 kg object initially is traveling to the left with a velocity of 4.00 m/s. If a 5.00 N force acts to the right for 1.80 seconds, what is the final velocity of the object?

6. A particle begins at rest and moves in a straight line so that one second later it has ended at rest one metre away. Show that at some point the magnitude of acceleration of the particle was at least 4.

7. A conveyor will utilize a motor-powered mechanical arm to exert an average force of 890 N to push large crates a distance of 12 meters in 22 seconds. Determine the power output required of such a motor.

8. A hoist operated by an electric motor has a mass of 500 kg. It raises a load of 300 kg vertically at a steady speed of 0.2m/s. Frictional resistance can be taken to be constant at 1200 N. What is the power required?

9. A car of mass 900 kg has an engine with power output of 42 kW. It can achieve a maximum speed of 120 km/h along the level.

What is the resistance to motion? If the maximum power and the resistance remained the same what would be the maximum speed the car could achieve up an incline of 1 in 40 along the slope?

10. A point object of mass 2 kg, moving in the plane with a bearing of 120° is subject to an impulse of 2 N secs towards the north. Find the subsequent velocity of this particle.

Problem Set 9 Conics

A *conic* C is the locus of points $P(r,\theta)$ satisfying the equation $OP = ePN$ where $e > 0$ is a constant known as the *eccentricity*, O is a point known as a *focus* and N is the *directrix*, which is a line with equation $N : x = k \neq 0$.

1. Show that the polar equation of C is given by

$$r = \frac{ke}{1 + e\cos\theta} \text{ or } r = \tfrac{ke}{1-e\cos\theta}$$

according as $k > 0$ or $k < 0$.

2. Find the eccentricity and directrix of the following conic and also find its equation in cartesian coordinates: $r = \frac{25}{10-10\cos\theta}$.

3. Consider the conic equation with $k < 0$ and with $e = 1$. Show that the change of variable $x = x' - \frac{k}{2}$ allows the equation to be written in the form $y^2 = 4ax'$ and find the focus and directrix of this parabola in $x'y$-coordinates.

4. Suppose now that the directrix is a line N whose normals make an angle θ_0 with the polar axis. Show that the equation of the conic is now

$$r = \frac{ke}{1 + e\cos(\theta - \theta_0)}$$

where k is the distance from O to the line N.

5. Show that the locus of points $P(x, y)$ that have the property that $|PF_1|+|PF_2| = 2a$ where $F_1 = (-c, 0)$ and $F_2 = (c, 0)$ is an ellipse and give its centre and the lengths of its axes.

6. Show that the locus of points $P(x, y)$ that have the property that $|PF_1|-|PF_2| = 2a$ where $F_1 = (-c, 0)$ and $F_2 = (c, 0)$ is a hyperbola give its centre and the equations of its asymptotes.

7. In general, show that the polar conic equation with $k < 0$ and with $e \neq 1$ can be written in the form

$$\frac{(x-c)^2}{a^2} \pm \frac{y^2}{b^2} = 1,$$

with the equation representing an ellipse if $e < 1$ and a hyperbola if $e > 1$.

8. Consider a planet the orbit of which has the Sun at a focus and whose equation is:

$$r = \frac{l}{1 + e\cos\theta},\ 0 < e < 1,\ l = ke > 0.$$

Find the furthest distance r_A and closest distance, r_P of the planet to the Sun (the so-called *aphelion* and *perihelion* distances) and the corresponding values of θ. Show that l, the *semi-latus rectum*, is the length of the vertical from the major axis of the ellipse from the focus to the curve itself.

9. Following on from the previous question, show that $a = \frac{ke}{1-e^2} = \frac{r_A}{1+e}$.

10. In a similar way show that semi-minor axis b of the ellipse of Question 8 satisfies $b = a\sqrt{1-e^2}$.

Problem Set 10 Kepler's Laws of Orbits

1. Show that for an object subject only to a central force from a point (acting only in the radial direction, such as gravity) $h = r^2\dot{\theta}$ is constant.

Kepler's Second Law

2. Show, using Question 1, that for a planet P orbiting the Sun O, the area swept out by the ray OP during the time interval $[t_1, t_2]$ is proportional to $t_2 - t_1$.

3. Use the law of gravity: $F(r) = -\frac{\gamma Mm}{r^2}$, where M and m are the respective masses of the Sun and a planet or asteroid to deduce that r satisfies the second order differential equation:

$$-\frac{\gamma M}{r^2} = \ddot{r} - \frac{h^2}{r^3}.$$

4. Show that, in these circumstances, we have the equality $\frac{d}{dt} = \frac{h}{r^2}\frac{d}{d\theta}$ and so the equation of Question 3 takes the form:

$$-\frac{\gamma M}{h^2} = \frac{d}{d\theta}(\frac{1}{r^2}\frac{dr}{d\theta}) - \frac{1}{r}.$$

5. Apply the substitution $u = \frac{1}{r}$ to the differential equation of Question 4 to recover a second order linear differential equation in u as a function of θ. Show that the general solution takes the form:$u = A\cos(\theta - \theta_0) + \frac{\gamma M}{h^2}$ and so

$$r = \frac{\frac{h^2}{\gamma M}}{1 + \frac{h^2 A}{\gamma M}\cos(\theta - \theta)},$$

a conic section with eccentricity $e = \frac{h^2 A}{\gamma M}$.

6. Show that the period T (year) of a planet in an elliptical orbit with semi-major axes a and b is given by $T = \frac{2\pi ab}{h}$.

7. The *gravitational potential* V of a unit mass m at a distance r from a mass M is the work done by gravity as the mass m moves from r to infinity. Show this is given by $V = -\frac{\gamma Mm}{r}$.

8. The *total energy* E of an orbiting mass m is $E = V + P$ (potential plus kinetic energy). By evaluating this at perihelion show that

$$E = \frac{\gamma Mm}{2l}(e^2 - 1).$$

9. Deduce from Question 8 that $E < 0$, $E = 0$, or $E > 0$ according as the orbit of the planet is elliptica, parabolic or hyperbolic and show that the planet will escape the solar system when orbiting at speed v at a distance r from the Sun if and only if

$$v \geq \sqrt{\frac{2\gamma M}{r}}.$$

10. Use Questions 5 and 6 together with Questions 9 and 10 of Set 9 to deduce *Kepler's Third Law* for a planet in an ellliptical orbit.

$$T = \frac{2\pi}{\sqrt{\gamma M}} \cdot a^{\frac{3}{2}}.$$

Hints for Problems

Problem Set 1

1 & 2 Use the $d = vt$ to write down relevant equations and be careful with the units you work with.

4. Avoid the infinite geometric series and just focus on the fly.

5. Use Pythagoras and work with time derivatives of the ends of the ladder.

8. Use $A \cos x + B \sin x = R \cos(x - \alpha)$.

9. Open the glass out and treat it as a pair of rectangles, corresponding to the inside and outside.

10. Try to exploit the symmetry in the problem if you want a quick answer, but there is always the infinite geometric series approach.

Problem Set 2

4. Work with the average speed.

6. At the top of the toss, $v = 0$.

9. Apply the formula of Question 8 and differentiate to get y'.

10. Again apply Question 8, substitute all the information given and solve.

Problem Set 3

1. Newton's (second) law: $F = ma$.

7. Write down Newton's law for the before and after circumstances of the balloon.

8. Write down an equation for the aircraft's thrust and integrate.

9. An average speed argument works well here.

10. Solve the quadratic arising from equating the positions of the two vehicles.

Problem Set 4

2. Write $\omega = \sqrt{\frac{\lambda}{ml}}$ and solve in the standard fashion.

10. Apply the results of the two previous problems.

Problem Set 5

1. Equate the gain in kinetic energy to the loss of potential energy, the expression for which has two parts corresponding to the portions of the chain that initially overhang the table and the part that passes over the edge.

2. The net downward force on the chain is that of gravity acting on the overhang.

3. A separable equation; substitute $x = \cosh y$ in the integral that arises.

4. Use Newton's 2nd Law and the chain rule.

6. Measure Potential energy from the lowest point and use that conservation of energy.

8. Resolve the horizontal and vertical components of the tension in the bob so that the forward acceleration of the bob matches that of the vehicle.

Problem Set 6

8 - 10. Apply Question 4.

Problem Set 7

9. Equate the centripetal and gravitational forces.

Problem Set 8

1. Write down equations for conservation of momentum and kinetic energy.

6. Let p denote a point in the time interval $[0, 1]$ where the particle is $\frac{1}{2}$ m from its initial points. Show by integration that if $p \leq \frac{1}{2}$ then $a \geq 4$ for some point in the interval $[0, p]$ while if $p \geq \frac{1}{2}$ then the particle decelerates at a rate of at least 4 m/sec^2 at some point in the interval $[1 - p, 1]$.

9. Find the resistance first, then add to that the component of gravity down the slope and use $v = \frac{P}{F}$.

Problem Set 9

1. & 4. A good diagram will help with the trigonometry in these questions.

2. Remember that $r^2 = x^2 + y^2$ and that $x = r\cos\theta$, $y = r\sin\theta$.

8 & 9 What values of θ will maximize and minimize the orbital distance r?

10. Express in terms of a and e the parameter k, the distance CO from the centre of the ellipse to the focus, and BO then distance from the top of the ellipse to the centre.

Problem Set 10

1. Differentiate $r^2\dot{\theta} = h$ and show that this is equivalent to the condition that the radial component of acceleration is zero.

2. The area swept out is $\frac{1}{2}\int_{\theta(t_2)}^{\theta(t_2)} r^2\,d\theta$. Now use the Chain rule and Question 1.

3. Equate the radial and gravitational forces acting on the mass m and use Question 1 to express $r\dot{\theta}^2$ in terms of h and r.

6. Calculate the area of the ellipse (πab) as a polar integral, change the variable of integration to time, and use Question 1.

7. At perihelion, the radial component of acceleration is 0 so that the corresponding velocity v_P satisfies $v_P = r\dot{\theta}$ and $h = r_P v_P$.

10. Show that $h^2 = \gamma M l$ and $l = a(1 - e^2)$.

Answers to the Problems

Problem Set 1

1. 1 mile. 2. $29\frac{1}{6}$km. 3. 10, 20. 4. 50 miles. 5. $-\frac{8}{3}$ m/sec. 6. $\frac{\sqrt{3}}{2}c$. 7. $-n^2x(t)$. 8. $\pm\sqrt{A^2+B^2}$; max first occurs at $\arctan(\frac{B}{A})$, the min first occurs at $\arctan\left(\frac{B}{A}\right)+\pi$. 3. 9.5inches. 10. $\frac{12}{11}$ hours.

Problem Set 2

1. $v^2 = u^2 + 2as$. 2. $2\cdot 5$m/sec^2. 3. 10m/sec. 4. $48\cdot 35$m. 5. $3\cdot 02$m. 6. $1\cdot 57$sec. 7. $44\cdot 1$m. 8. $x\tan\alpha - \frac{gx^2\sec^2\alpha}{2V^2}$. 9. $\arctan^{-1}\left(\frac{1}{9}\right)$ below horizontal. 10. $\arctan\left(\frac{2}{3}\right)$, arctan(2).

Problem Set 3

1. $F = 2\times 10^3\times 2\cdot 5 = 5\times 10^3$N. 2. $2\times 10^3\times 1 = 2\times 10^3$N. 3. $m(g+f)$. 4.$m_1(g+f)$. 5. $\frac{m_1f+m_2g}{m_1-m_2}$. 6. $233\cdot 9$ m 7. $m = \frac{M(f_1+f_2)}{g+f_2}$. 8. $\frac{50}{3}$sec. 9. $30\cdot 37$ m. 10. Yes.

Problem Set 4

1. $\frac{mgl}{x_0}$. 2. $C\cos(\omega t+\varepsilon)+x_0$. 3. $(a-x_0)\cos(\omega t)+x_0$, where $\omega = \sqrt{\frac{\lambda}{ml}}$. 4. $\frac{\omega}{2\pi}$, $\frac{2\pi}{\omega}$, a. 5. $v = \pm\sqrt{2gx-\omega^2x^2}$. 9. $\frac{\lambda a^2}{2l}$. 10. $1\cdot 96\times 10^4$N.

Problem Set 5

5. $a = \frac{m_2-m_1}{m_1+m_2}g$, $T = \frac{2m_1m_2}{m_1+m_2}g$. 6. $\sqrt{2ga(1-\cos\theta}$. 7. $\cos^{-1}(\frac{2}{3})$. 9. $\arctan^{-1}(\frac{a}{g})$.

Problem Set 6

1. $\mathbf{i} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\theta}$ and $\mathbf{j} = \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\theta}$.2. $\dot{\hat{\mathbf{r}}} = \dot{\theta}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = \dot{\theta}\hat{\theta}$ $\dot{\hat{\theta}} = \dot{\theta}(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j}) = -\dot{\theta}\hat{\mathbf{r}}$.3. $\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}$. 4. $(\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$. 5. $\frac{v}{a\sqrt{5+4\cos\theta}}$. 6. $r\omega^2 = \frac{v^2}{r}$. 7. 4.22×10^7 m. 8. $\mathbf{a} = -l\dot{\theta}^2\hat{\mathbf{r}} + l\ddot{\theta}\hat{\theta}$ $\mathbf{F} = -T\hat{\mathbf{r}} + mg\cos\theta\hat{\mathbf{r}} - mg\sin\theta\hat{\theta}$. 9. $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$. $\theta(t) = A\cos(\omega t + \alpha)$ where $\omega = \sqrt{\frac{g}{l}}$. The period is $2\pi\sqrt{\frac{l}{g}}$. 10. $\cos\theta = \frac{g}{\omega^2 l}$.

Problem Set 7

1. 15 : 00 hrs. 2. 372 : 1. 3. 51,480,000 4. 366. 5. 2.828. 6. $9 \cdot 6$ 7. $r = 4.22 \times 10^7$ m. 9. 4.22×10^7 m.

Problem Set 8

1. 3. 3. 8kg. 4. 1000N. 5. 1 m/sec. 7. 480 watts. 8.$1 \cdot 81$ kw. 9. 102 km/hr. 10. $v = 2 \cdot 65$, $101 \cdot 2°$ East of North.

Problem Set 9

2. $e = 1$, $k = 2 \cdot 5$, $x = \frac{1}{5}y^2 - \frac{5}{4}$. 3. $N : x' = -a$, $O : (a, 0)$. 5. $(0, 0)$ a and $\sqrt{a^2 - c^2}$. 6. $(0, 0)$, $y = \pm\frac{\sqrt{a^2-c^2}}{a}x$. 8. $r_A = \frac{l}{1-e}$, $\theta = \pi$, $r_P = \frac{l}{1+e}$, $\theta = 0$. 10. $a = \frac{ke}{1-e^2}$, $b = a\sqrt{1 - e^2}$.

Problem Set 10

7. $V = -\frac{\gamma M}{r}$.