Mathematics 101 Algebra & Complex Numbers

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This introductory module covers basic algebraic facts and techniques along with introduction to the so-called *complex numbers.* As a general rule, key topics will be italicised when they first appear and will thereby act as a search prompt that will allow you to research the topic to the extent you need to in order to be able to tackle the problems in each module.

The specific topics that you will meet here in the order they arise include, *rationalising the denominator*, *indices and logarithms*, *laws of algebra*, *roots of polynomials*, the *Remainder* and *Factor theorems, imaginary* and *complex numbers* in *cartesian* and *polar form*, *equations in polar co-ordinates*, and *arithmetic* and *geometric series.*

Problem Set 1 Algebra

1. Express with a *rational denominator*

$$\frac{1}{3\sqrt{5}-\sqrt{7}}.$$

2. Write $1 - x^2 + 3x$ in the form $a(x-b)^2 + c$.

3. The graph of $y = x^2 - 6x + 13$ can be obtained from that of $y = x^2$ by translating the graph of the latter a units to the right and b units up. Find the value of a and of b.

4. What number most exceeds its own square?

5. That

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

is a consequence of which law of algebra?

6. If $S = 2\pi r(h+r)$ and $V = \pi r^2 h$, find a formula for V in terms of S and r only.

7. A solid cylinder of gold, height 14cm and base radius 8cm, is melted down and cast as a solid cube. Calculate, to the nearest millimetre, the length of an edge of that cube.

8. If $f(x) = x^2 + 5x + 2$ and $g(x) = (3x+2)(x^2+x+1)$, find the values of x for which $f(x) > g(x)$.

9. What is the remainder when

$$10x^{75} - 8x^{65} + 6x^{45} + 4x^{32} - 2x^{15} + 5$$

is divided by $x - 1$?

10. Find the value of k such that $x + 2$ is a factor of

$$x^3 - kx^2 + 3x + 7k.$$

Problem Set 2 Inequalities

1. Find the least value of x such that

$$1 - 4x \leq -\frac{1}{2}x - 1.$$

2. Solve the inequality

$$-4 \leq \frac{4-2x}{3} < 4.$$

3. Solve the double inequality

$$4x - 2 < x + 8 < 9x + 1.$$

4. Sketch the region in the first quadrant satisfying the following inequalities, labelling the intersection points of boundary lines by their co-ordinates:

$$x + y \geq 3,\ y < 6,\ x \leq \frac{y+5}{4}.$$

5. Find all values of x for which

$$\frac{x+3}{x-1} \geq -2.$$

6. Find all values of x such that

$$\frac{x-1}{1-3x} > 7.$$

7. Find all values of x such that
(a) $|2x+1| \leq 5$;
(b) $|1-4x| > 9$.

8(a) Given that $|x-2| \leq 3$ and $|y+2| < 1$, what is the set of possible values of $x+y$?
(b) Given that $|x+2| \leq 5$ and $|-2y-2| < 8$, find the set of values of xy.

9. Show that the equation:

$$|ax + b| = |cx + d||$$

generally has two solutions. Determine when the solution is unique.

10. Shade the region in the cartesian plane where holds the inequality

$$\frac{1}{x} \leq \frac{1}{y}.$$

Problem Set 3 Powers and Logarithms

1. Solve

$$2^{x^2+5x} = \frac{1}{16}.$$

2. I deposit £4,000 in a bond that compounds interest annually. After five years my account is worth £6,740.23. What is the annual rate of interest?

3. A company bought computing equipment for £10,000 and sold it three years later for half that amount. Calculate the annual rate of *depreciation*, giving your answer correct to one decimal place.

4. Solve

$$1 + \log_2(\log_2 x) = 0.$$

5. Solve the equation:

$$(\log_2 x)^2 + \log_2 x^3 - 10 = 0.$$

6. Find all real values of x for which

$$\log_3 x - 2\log_x 3 = 1$$

.

7. Solve for x:

$$\log_a(4x) - 3\log_a(x^2) = \log_a(128).$$

8. Solve

$$\log_3(\log_8 x) + 1 = 0.$$

9. Simplify the expression $9^{2\log_3 2}$.

10. Find all six integer solutions to

$$(x^2 - 5x + 5)^{(x^2-11x+30)} = 1.$$

Problem Set 4 Equations

1. Find the equation of the line through $(1, 2)$, perpendicular to the line $2x + y + 1 = 0$, giving your answer in slope-intercept form $y = mx + c$.

2. Find all real values of x such that

$$|x| + |-2x| = 1.$$

3. Solve

$$3 - x + \sqrt{3 - x} = 0.$$

4. Solve

$$\sqrt{2 + x} + x = 10.$$

5. For precisely which real numbers a and b is it true that

$$\sqrt{a^2 + b^2} = a + b?$$

6. Find the equation of the parabola $y = ax^2 + bx + c$ that passes through the three points $(1, -6), (-3, 30), (0, -3)$.

7. Find the equation of the line that forms the locus of all points equidistant from $(-1, 2)$ and $(-5, -3)$.

8. Find the equation for the inverse of the function $y = e^{1-x}$.

9. Give two examples of non-identity functions that equal their own inverses.

10. A bottle of whiskey lasts a man 12 days. When his friend visits him, it lasts both of them 8 days. How long would such a bottle of whiskey last his friend alone?

Problem Set 5 Algebra II

1. *Without solving the corresponding equation*, find the sum of the three roots of the polynomial

$$2x^3 - 8x^2 + 20x - 19.$$

2. Let α and β be the roots of the equation

$$x^2 + 3x + 4 = 0.$$

Without solving the equation, find the value of $\alpha^3 + \beta^3$.

3. The roots of the equation $x^2 + 2x + 3 = 0$ are α and β. Without solving the equation, find a quadratic equation the roots of which are $1 - \frac{1}{\alpha}$, $1 - \frac{1}{\beta}$.

4. Function composition satisfies the law $f \circ (g \circ h) = (f \circ g) \circ h$. Which law of algebra is this?

5. Find the equations of the *asymptotes* of the function:

$$y = \frac{2x^2 + x - 4}{x - 1}.$$

6. Find all solutions to

$$2x^4 + x^2 = 6.$$

7. Factorize the foloowing polynomial into a product of linear factors:

$$x^3 + 5x^2 - 12.$$

8. Find where the following line meets the unit circle centred at the origin.

$$4x - y - 4 = 0.$$

9. Find the partial fraction decomposition of

$$\frac{1}{1 + x^3}.$$

.

10. Express $\sqrt{3 - \sqrt{5}}$ in the form $\frac{\sqrt{a} - \sqrt{b}}{c}$ for some integers a,b and c.

Problem Set 6 Complex numbers

1. Carry out the division of complex numbers

$$\frac{15 + 16i}{2 + 3i}.$$

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2. Find $|z^{10}|$, where $z = 1 + 7i$.

3. Find the modulus of the complex number

$$z = \frac{-8 \cdot 5 + 1 \cdot 72i}{-1 \cdot 72 - 8 \cdot 5i}.$$

.

4. Solve the equation

$$(7 + i)x - 3 = x + 22 - 2i.$$

5. Solve $z^2 = 3 - 4i$ by using both cartesian and polar co-ordinates.

6. Solve for complex z the equation

$$|z| - z = 1 - 2i.$$

7. Sketch in the complex plane the locus of a point P representing the complex number z, where $|z - 1| = |z - 3i|$.

8. Determine the region in the complex plane defined by

$$\text{Arg}(2 + iz) \leq \tfrac{\pi}{4}.$$

9. Find all complex z such that $z^3 = z\overline{z}$.

10. Given that

$$f(z) = (x^2 - y^2 - x) + (2xy - y + 1)i,$$

where $z = x + iy$, express $f(z)$ in terms of z only.

Problem Set 7 Polar co-ordinates

1. Find the polar co-ordinates of the point P, the rectangular co-ordinates of which are $(-2, 2\sqrt{3})$.

2. Sketch the graph of the curve the polar equation of which is

$$r = 1 + 2\cos\theta.$$

3. Repeat Question 2 for the four-leaf rose:

$$r = 5\sin 2\theta.$$

4. Sketch the curve, the polar equation of which is

$$r^2 = 2a^2\cos 2\theta.$$

5. By first multiplying both sides by r^2, find the equation of the *lemniscate* of Question 4 in cartesian form.

6. Find the equation of the circle with centre $(1, 0)$ and radius 1 in both cartesian and polar forms.

7. Find the polar equation of the line with cartesian equation $y = 1 + 2x$.

8. Find $[2(\cos 15^\circ + i\sin 15^\circ)]^4$.

9. Find the square root of $-i$ that lies in the 4th quadrant, giving your answer in cartesian form.

10. Find the three cube roots of -8, giving your answer in cartesian form.

Problem Set 8 Equations II

1. The following problem dates back to the 16th century, if not earlier.

Twenty people pay twenty pence to visit the Lincoln Fair. If each man pays threepence, each woman tuppence, and each child a halfpenny, how many men, how many women, and how many children went to the fair?

2. Three comrades A, B and C from the ECP (Extreme Centre Party) leaflet their street each month. When A and B do it together it takes 15 minutes, when B and C work together it takes only 10 minutes while A and C require 12 minutes for the job. How long would it take if all three worked together?

3. A point (x, y) on a curve is called *rational* if both x and y are rational numbers. For example, the point $(0,1)$ on the graph of $x^3 - 3x^2 + 3x + 1 = y^2$ is a rational point. Find another rational point by locating the intersection of the curve with its own tangent at the point $(0,1)$.

4. This technique for finding one rational solution to an equation from a given one is due to *Diophantus* whose life is extremely obscure—we only know for certain that he lived between AD 150 and AD 350. We do know however how long he lived, for someone left behind this riddle.

Diophantus lived $\frac{1}{6}$ of his life as a child, $\frac{1}{12}$ as a youth, $\frac{1}{7}$ as a bachelor, and five years after his marriage a son was born who lived only half as long as did his father and died four years before him. How long was the life of Diophantus?

5. Solve the simultaneous equations:

$$\frac{1}{x} - \frac{3}{y} = 2, \quad \frac{2}{x} + \frac{1}{y} = 3.$$

6. Find the value of x given that $x + 3y + 3z = 1$ and $6x - 2y - 2z = 4$.

7. *Sam Loyd's River Boat Problem* Two ferries set sail from opposite sides of the river at constant but different speeds. They first pass one another 720 yards from one bank. They reach their destinations whereupon passengers disembark and new ones get on in a ten minute changeover period. They then return to their original berths and pass for a second time 400 yards from the second bank while doing so. How wide is the river?

8. Continuing with Loyd's problem, what is the ratio of the speed of the slower to the faster ferry?

9. A man is running for a bus at speed v. As he gets within d metres, the bus begins to move away from him, accelerating from rest with constant acceleration a. What is the maximum value of d that allows him to catch the bus?

10. A ship is twice as old as its boiler was when the ship was as old as the boiler is now. What is the ratio of the ship's age to that of its boiler?

Problem Set 9 Sequences and series

1. The third term in an *arithmetic sequence* is 4 and the *common difference* of the sequence is 3. Find the nth term of the sequence.

2. Find the sum of the first $2n$ terms of the sequence in Question 1.

3. Evaluate

$$2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{1024},$$

giving your answer as a *mixed number*, that is an integer followed by a fraction between 0 and 1.

4. Given that $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$, for $|x| < 1$, evaluate:

$$\sum_{n=1}^{\infty} \frac{n-2}{3^n}.$$

5. An arithmetic progression has its mth term equal to n and its nth term equal to m (for some specific $m < n$). Find the value of its $(m+n)$th term.

6. Find the sum S_n of the series $r + r^3 + r^5 + \cdots + r^{2n-1}$ and also find $\lim_{n\to\infty} S_n$, where $r = \frac{1}{2}$.

7. Find $1 + 2 + \cdots + n$ by summing the identity:

$$(m+1)^2 - m^2 = 2m + 1;$$

from $m = 1$ to n.

8. Similarly find $1^2 + 2^2 + \cdots + n^2$ using the identity:

$$(m+1)^3 - m^3 = 3m^2 + 3m + 1.$$

9. By considering a suitable binomial expression $(a+b)^n$, find the value of the sum

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}.$$

10. By using the *partial fraction decomposition* of the summand, find:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

Problem Set 10 Equations III

1. What temperature has the same value in both Celsius and Farenheit?

2. There are four black cows and three brown cows. These give the same amount of milk in five days as three black cows and five brown cows in four days. Which are the better milk producers, the black or the brown cows?

3. (From the Chinese manuscript: *Nine Chapters on the Mathematcial Art*, cf. 250BC) A good runner covers 100 paces in the time a poor runner covers 60 paces. The good runner sets off in pursuit of the poor runner who has a 100 paces start. How many paces does it take for the strong runner to catch the slower one?

4. The film *The Interview* grossed \$15 million income on the weekend of its release. The film cost \$6 to rent, \$15 to buy and two million copies were distributed overall. How many copies were rented and how many copies were bought?

5. Find all solutions to $0 = 1 + x + x^2 + x^3$.

6. Find all solutions to $1 = x^2 + x^3 + x^4 + \cdots$.

7. *Ancient Hindu Riddle* Three travellers stopped at an inn for supper. The innkeeper could only offer them baked potatoes. While the potatoes were baking the travellers fell asleep. Soon one awoke to find a dish of baked potatoes. He took one third of them and went back to sleep. Then a second traveller awoke, took a third of the remaining potatoes, and again fell asleep. Finally the third one awoke, and did the same as his two travelling companions. When the innkeeper cleared up, there were eight potatoes remaining. How many were there initially?

Solution of the cubic.

8. First find the value of t such that the substitution $x = y + t$ transforms the equation $x^3 + ax^2 + bx + c = 0$ into that of a *depressed cubic:* $y^3 + dy + e = 0$ (a, b, c, d, e are constants).

9. Now for the depressed cubic of Question 8, find the substitution $y = v + \frac{s}{v}$ that transforms the equation to one of the form $v^3 + \frac{f}{v^3} + g = 0$ and hence reduce the problem to that of a quadratic equation in $z = v^3$.

10. Apply this technique to find a real root of $p(x) = x^3 - 3x^2 + 6x + 8$.

Hints for Problems

Problem Set 1

1. Multiply top and bottom by the conjugate surd in order to rationalise the denominator.

4. This is just a tricky way of asking you to maximize $y = x - x^2$.

5. You may need to look up the name of the Law involved but first decide exactly what you are doing as you pass from one side of the equation to the other.

8. Subtract $f(x)$ from $g(x)$ to get an expression that you want to keep negative. Factorize it and then the solution should be clear.

9 & 10. Remember the *Remainde*r and *Factor Theorems*: a polynomial $p(x)$ leaves the remainder $p(a)$ when divided by $x - a$ and so $x - a$ is a factor of $p(x)$ iff $p(a) = 0$. Certainly don't do the divisions involved!

Problem Set 2

4. Easy enough but be careful to label the figure properly, which of course requires you to find the co-ordinates where boundary lines meet. Boundary lines that are not part of the region should appear hatched.

5 & 6. Best bring everything over to one side and put over a common denominator. Try to avoid multiplying both sides by an expression of indeterminant sign.

7 & 8. Remember that $|x - a| \leq b$ can be expressed as $-b \leq x - a \leq b$.

Problem Set 3

1. Write it all in terms of powers of 2 and then equate exponents.

2 & 3. If the annual rate is $x\%$ then each year your principal is multiplied by the factor $\left(1 + \frac{x}{100}\right)$.

Remember your log laws and for 5, 6 & 10 how to change bases:

$$\log_b x = (\log_a x)(\log_b a).$$

Problem Set 4

1. The product of the gradients of perpendicular lines is -1.
2. Use $|ab| = |a| \cdot |b|$ to simplify.
3. Beware! Squaring can drag in extraneous roots.
7. Work algebraically with the squares of the distances. Or think geometrically.
10. Work in units of bottles/day.

Problem Set 5

1-3. Expanding $(x - r_1) \cdots (x - r_n)$ will give various sum of products of the roots r_i in terms of the coefficients of the polynomial.

5. There are two aymptotes: the oblique one is found by polynomial divison.
7. Apply the Rational root theorem to find an integer root, which gives the first linear factor.
9. The *irreducible quadratic factor* will require a fraction with numerator of the form $Ax + B$.
10. Square both sides and equate rational and irrational parts.

Problem Set 6

1. Multiply top and bottom by the *conjugate* $\overline{z} = x - iy$ of the denominator $z = x + iy$.
2. The modulus of a power is the power of the modulus.
5. Cartesian or polar form will work, but there can only be two solutions.
6. Equate real and imaginary parts.
8. $\theta =$Arg$z$ is in the range $-\pi < \theta \leq \pi$. Try to identify the boundary of the set.
9. Use polar form.
10. Substitute $x = \frac{1}{2}(z + \overline{z})$ and $y = \frac{1}{2i}(z - \overline{z})$.

Problem Set 7

1. You want polar co-ordinates (r, θ), where r is the *distance* from O to P and θ is the *angle, in radians,* measured anti-clockwise from the positive x-axis to the ray OP. (Draw a diagram.)

2. This is a graph in polar co-ordinates. Remember that cosine is an *even function* so this graph will be symmetric with respect to the polar (x-) axis. This is curve is called a *limacon*, and manages to cross itself at the origin.

7. Write the tan of the polar angle in terms of x and convert to full polars.

8. *DeMoivre's Theorem* says that $(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$, and then convert back to cartesian form.

10. For roots, De Moivre's Theorem says that the nth roots of $re^{i\theta}$ form the set $\{r^{\frac{1}{n}}e^{i\frac{\theta}{n}},\ r^{\frac{1}{n}}e^{i(\frac{\theta+2\pi}{n})}, \cdots, r^{\frac{1}{n}}e^{i(\frac{\theta+2k\pi}{n})}, \cdots, r^{\frac{1}{n}}e^{i(\frac{\theta+2(n-1)\pi}{n})}\}$.

Problem Set 8

1. Write down two equations and remember that people come in positive integer numbers.

4. Summarise all the information in a single equation and solve it.

5. The obvious substitutions make the problem easy.

9. Quite easy to generate a quadratic equation but the answer to the question is not its roots but lies in the nature of its discriminant.

10. Introduce symbols, name them precisely, then write down an equation.

Problem Set 9

An *arithmetic progression* with first term a and *common difference* d has $t_n = a + (n-1)d$ and $\sum_{k=1}^{n} t_k = na + \frac{d}{2}n(n-1)$. A *geometric progression* with initial term a and *common ratio* r has $t_n = ar^{n-1}$ and $\sum_{k=1}^{n} t_k = a\frac{1-r^n}{1-r}$; the infinite sum is $\frac{a}{1-r}$ iff $|r| < 1$.

4. Can split it into two series, one of which is geometric and the given formula deals with the other. Alternatively, by re-indexing the sum and taking out two terms separately, the formula suffices.

6. Be careful in identify the initial term and the common ratio.

Problem Set 10

1. You need the conversion formula $F = \frac{9}{5}C + 32$.
2. Write all the information as a single equation and simplify.
4. Work in unit of millions and express as a pair of simultaneous equations.
5. Think geometric progression.
6. Sum the series but remember to be a solution that series must converge.
7. As with Question 5.

Answers to the Problems

Problem Set 1

1. $\frac{3\sqrt{5}+\sqrt{7}}{38}$. 2. $a=-1, b=\frac{3}{2}, c=\frac{13}{4}$. 3. $a=3, b=4$. 4. $\frac{1}{2}$. 5. *Distributive Law.* 6. $\frac{r}{2}\left(S-2\pi r^2\right)=\frac{1}{2}rS-\pi r^3$. 7. 141mm. 8. $x<-\frac{4}{3}$. 9. 15. 10. $k=\frac{14}{3}$.

Problem Set 2

1. $\frac{4}{7}$. 2.$-4<x<8$. 3. $\frac{7}{8}<x<\frac{10}{3}$. 5. $(-\infty,-\frac{1}{3}]\cup(1,\infty)$. 6. $\frac{1}{3}<x<\frac{4}{11}$. 7. $-3\le x\le 2$. 8(a) $-4<x+y<4$, 8(b). $-21<xy<35$.

Problem Set 3

1. $\{-1,-4\}$. 2. 11% . 3. $20\cdot 6\%$ 4. $\{4,\frac{1}{32}\}$. 5. $\sqrt{2}$. 6. $\{\frac{1}{3},9\}$. 7. $\frac{1}{2\sqrt[5]{a}}$. 8. 2 . 9. 16 . 10. $\{1,2,3,4,5,6\}$.

Problem Set 4

1. $y=\frac{1}{2}x+\frac{3}{2}$. 2. $\pm\frac{1}{3}$. 3. 3. 4. 7. 5. At least one of a and b equals 0 <u>and</u> $a,b\ge 0$. 6. $y=2x^2-5x-3$. 7. $y-\frac{4}{5}x-\frac{29}{10}$. 8. $y=1-\ln x$. 9. $y=a-x$, $y=\frac{k}{x}$. 10. 24 days.

Problem Set 5

1. 4. 2. 9. 3. $3x^2-8x+6=0$. 4. The Associative law. 5. $x=1$ and $y=2x+3$. 6. $x=\pm\sqrt{2}i$ or $x=\pm\frac{\sqrt{6}}{2}$. 7. $(x+2)(x+\frac{3+\sqrt{33}}{2})(x+\frac{3-\sqrt{33}}{2})$. 8. $(1,0)$ or $(\frac{15}{17},-\frac{8}{17})$. 9. $\frac{1}{3(1+x)}+\frac{2-x}{3(x^2-x+1)}$. 10. $\frac{\sqrt{10}-\sqrt{2}}{2}$.

Problem Set 6

1. $6-i$. 2. $312,5000,000$. 3. 1. 4. $4-i$. 5. $\pm(2-i)$. 6. $\frac{3}{2}+2i$. 7. $y=\frac{1}{3}x+\frac{4}{3}$. 9. $\{0,1,-\frac{1}{2}+\frac{\sqrt{3}}{2}i,\frac{1}{2}-\frac{\sqrt{3}}{2}i\}$.10. z^2-z+i .

Problem Set 7

1. $(4,\frac{2\pi}{3})$. 5. $r^4=2a^2(x^2-y^2)$. 6.$r=2\cos\theta$. 7. $r(\sin\theta-2\cos\theta)=1$. 8. $8+8\sqrt{3}i$. 9. $\pm\frac{1}{\sqrt{2}}(1-i)$. 10. $1+\sqrt{3}, i-2, -1+\sqrt{3}i$.

Problem Set 8

1. 14 children, 5 women and 1 man. 2. 8 minutes. 3. $\frac{21}{4},\frac{71}{8}\big)$. 4. 84 years. 5. $\big(\frac{7}{11},-7\big)$. 6. $\frac{7}{10}$. 7. 1 mile. 8. $9:13$. 9. $\frac{v^2}{2a}$. 10. $4:3$.

Problem Set 9

1. $3n-5$. 2. $6n^2-7n$. 3. $3\,\frac{1023}{1024}$. 4. $-\frac{1}{4}$. 5. 0. 6. $\frac{2}{3}$. 7. $\frac{1}{2}n(n+1)$. 8. $\frac{n}{6}(n+1)(2n+1)$. 9. 0. 10. 1.

Problem Set 10