Mathematics 107 Further Calculus & Differential equations

Professor Peter M. Higgins

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Synoposis of MA107

The new topics that you will meet in this module are the following. In Set 1, *Linear approximation* using differentials (in one variable), *Simpson's rule* for numerical integration by means of approximating parabolas, and the *Newton-Raphson* method for finding roots of equations through extrapolating from successive tangent points. Set 2 includes questions on *volumes of revolution.* Set 3 is on limits including use of *L'Hopital's rule.* Set 4 features applications of the one-dimensional chain rule. The integrations of Set 5 introduce recursive reduction formulas and *arc length* calculations.

Differential equations is the topic in Sets 6 and 7 and we work through the standard types they being, *separable equations*, *exact differential equations*, and use of *integrating factors* for the non-exact case. Set 7 begins with examples of *first order linear equations* followed by *second order linear differential equations* of the standard types.

Set 8 is on power series with the emphasis on derivation of *Taylor series* and use of bounds on the remainder term for truncated series.

Sets 9 and 10 involve functions of several variables. There are calculations involving *partial derivatives*, the *chain rule* for several variables, the *gradient function* and its properties, leading to problems in optimization. These include finding and classifying extreme points of functions of several variables and the use of *Lagrange multipliers* to find extreme points along curves and surfaces.

Problem Set I Approximations

The *linearization* $L(x)$ of a differentiable function $f(x)$ at $x = a$ is the linear function $L(x) = f(a) + (x - a)f'(a)$. The graph of $L(x)$ is the tangent to the graph of $f(x)$ at $(a, f(a))$ and $L(x)$ is the best linear approximation to $f(x)$ near $x = a$ in that it is the only linear function $L(x)$ that satisfies $L(a) = f(a)$ and $\lim_{h\to 0} \frac{f(a+h)-L(a+h)}{h} = 0$. This can be also be used in the form $f(a + \Delta x) \approx f(a) + f'(a)\Delta x$ to approximate the value of $f(a+\Delta x)$ for a small increment Δx of x.

1. Find the linearization of $f(x) = \frac{1}{x}$ at $x = \frac{1}{2}$.

2. A ball of ice melts so that its radius decreases from 5cm to $4 \cdot 92$cm. Use a linear approximation of the volume V of the ball to find approximately by how much the volume of the ball decreases.

3. Use the linearization of $f(x) = \sqrt{x}$ at $x = 25$ to find an approximate value for $\sqrt{26}$.

Simpson's Rule approximates $\int_a^b f(x)\,dx$ by a sum S_n based on partitioning $[a, b]$ into an *even number* n of equal segments, with endpoints $a = x_0, x_1, \cdots, x_n = b$ so that $x_i = a + ih$ $(0 \le i \le n)$, where $h = \frac{b-a}{n}$. Write $y_i = f(x_i)$; then $S_n = \frac{h}{3}\big(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{n-1} + y_n\big)$. Moreover if $f(x)$ has a 4th derivative $f^{(4)}(x)$ satisfying $|f^{(4)}(x)| \le K$ on $[a, b]$, then the error in Simpson's Rule is no more than $\frac{K(b-a)^5}{180n^4}$.

4. Calculate the S_8 approximation to $\int_1^2 \frac{dx}{x}$ and compare the answer to the exact value of $\ln 2$.

5. Show that for $f(x) = \frac{1}{x}$ we have $|f^{(4)}(x)| \le 24$ on $[1, 2]$.

6. Use your answer to Question 5 to calculate the error bound for S_8 from Question 4.

7. Use Simpson's Rule with $n = 4$ to estimate:

$$\int_0^1 5x^4\,dx.$$

Find an estimate of the error and compare with the exact result.

8. The *Newton-Raphson Method* for approximating a root of a function $y = f(x)$ begins with an initial approximate root x_0 and then calculates recursively the sequence of approximations:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Noting that $f(x) = x^3 - x - 1$ has a root in the interval $[1, 2]$, use $x_0 = 1 \cdot 5$ to calculate up to x_5, showing that $f(x_5) = 0 \cdot 00 \cdots$ to 10 decimal places.

9. Repeat Question 8 with a suitable $f(x)$ to find an approximate solution to $x^3 = \cos x$ by calculating up to x_5, using $x_0 = 1$ as the initial value.

10. By starting with *any* seed number and repeatedly pressing the 'cos' button on your calculator, solve the equation $x = \cos x$ (correct to four decimal places, working in radians, of course).

Problem Set 2 Integration

1. Find $\int \frac{dx}{1-x^2}$.

2. Evaluate the integral $\int_0^1 \frac{2\,dx}{\sqrt{1+4x^2}}$ using the substitution $x = \frac{1}{2}\sinh u$.

3. Find the volume generated by revolving about the x-axis the region bounded by the curves $y = \sinh x$, $y = \cosh x$ and the y-axis.

4. Find the asymptotes of the curve

$$y = \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}$$

by expressing y in the form $p(x) + \frac{q(x)}{x^2-2x-3}$, for linear polynomials $p(x)$ and $q(x)$.

5. Find the partial fraction decomposition of $\frac{q(x)}{x^2-2x-3}$, where $q(x)$ is as in Question 4.

6. Use Question 5 to find

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx$$

.

7. Evaluate the integral

$$\int_0^\pi \sqrt{\frac{1+\cos 2x}{2}}\,dx$$

.

8. Evaluate $\int_0^\pi \frac{\sin x}{2-\cos x}\,dx$.

9. Let $F(m) = \int_0^\infty x^{m-1}e^{-x}dx$ where $m \geq 2$ is an integer. Evaluate and simplify $\frac{F(m)}{F(m-1)}$ using integration by parts.

10. Find the volume swept out when the curve $y = \sqrt{x}$ $(0 \leq x \leq 4)$ is revolved around the x-axis.

Problem Set 3 Limits

Find the values of each of the following limits.

1.
$$\lim_{x\to\infty} \frac{3x^3 - 18x - 1}{-6x^3 + x^2}.$$

2.
$$\lim_{x\to 0} \frac{\sin 7x}{x}$$

3.
$$\lim_{x\to 0} \frac{1-\cos x}{x}$$

4.
$$\lim_{n\to\infty} (n(1+\frac{1}{n}) - n)$$

5.
$$\lim_{x\to 0} x \ln x$$

6.
$$\lim_{x\to 0} x^x$$

7.
$$\lim_{n\to\infty} n^{\frac{1}{n}}$$

8.
$$\lim_{n\to\infty} (1+\frac{1}{2n})^n$$

9.
$$\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n})$$

10.
$$\lim_{x\to 0^+} \frac{\sin\sqrt{x}}{x}.$$

Problem Set 4 Differentiation

1. Find the derivative of $y = \cosh^{-1} x$.

2. The radius of a soap bubble is increasing at the rate of 0.1cm/sec. How fast is the volume of the (spherical) bubble increasing when its radius is 0.8 cm?

3. Two lorries leave a depot, A travelling north at 40mph and B travelling east at 30mph. How fast is the distance between them increasing 6 minutes later?

4. At what rate is the area A of a rectangle increasing if its length $x = 15$m is increasing at 3 m/sec while its width $y = 6$m is increasing at 2m/sec?

5. Find the angle between the curves $y = x^3$ and $y = \sqrt{x}$ at the point $(1, 1)$.

6. Let $z = \sqrt{xy + y}$, $x = \cos t$, $y = \sin t$. Use the Chain Rule to find $\frac{dz}{dt}$ when $t = \frac{\pi}{2}$.

7. The respective distances u and v of an object and it image from a lens of focal length f are related by the equation $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$. An object is moved towards a lens of focal length 6cm at a speed of 4cm/sec. How fast is the image receding from the lens when the object is 5cm from the lens?

8. Find the length of the shortest ladder that can extend from a vertical wall, over a fence 2m high located 1m away from the wall, to a point on the ground outside the fence.

9. A lighthouse beacon is 4km from a straight shoreline and makes one revolution every 10 seconds. How fast is the beam moving along the shore when the ray makes an angle of 45° with the shoreline?

10. The function $f(x) = e^x + x$ being one-to-one and differentiable has a differentiable inverse $f^{-1}(x)$. Find $\frac{df^{-1}}{dx}|_{\ln 2}$.

Problem Set 5 Integration II

1. Let I_n denote the integral $\int x^n e^x \, dx$. By integrating by parts, express I_n in terms of I_{n-1}.

2. Use the formula that you derived in Question 1 to write down $\int x^3 e^x \, dx$.

3. Establish the reduction formula:

$$\int (\sin^{-1} x)^n dx = x(\sin^{-1} x)^n + n\sqrt{1-x^2}(\sin^{-1} x)^{n-1} - n(n-1)\int (\sin^{-1} x)^{n-2} dx.$$

4. Use Question 3 to find $\int (\sin^{-1} x)^2 dx$.

5. Find the *arc length* of the parabola $y = x^2$ from the origin to the point $(1, 1)$.

6. Find the length of the helix $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}$ from $t = 0$ to $t = \frac{\pi}{2}$.

7. Find the value of the *improper integral*

$$\int_1^4 \frac{dx}{(x-2)^{2/3}}.$$

8. Show that

$$\int_{-a}^{a} f(-x)\, dx = \int_{-a}^{a} f(x)\, dx.$$

9. Use Question 8 to find

$$\int_{-1}^{1} \frac{dx}{1 + x^5 + \sqrt{1 + x^{10}}}.$$

10. By use of a clever substitution evaluate the integral:

$$\int_0^\infty \frac{\ln x}{x^2 + 2x + 4}\, dx.$$

Problem Set 6 Separable and exact differential equations

Solve each of the following differential equations.

1.
$$\frac{dy}{dt} = -\frac{y^2}{t},\ t \neq 0$$
.

2.
$$\frac{dy}{dx} = (1+y^2)e^x$$
.

3. Find all functions whose derivatives equal their own reciprocals.

4. Find all functions which, when differentiated, equal their own square.

5. For the differential equation
$$\frac{dy}{dx} = \frac{y-4x}{x-y}$$
introduce the new variable $v = \frac{y}{x}$ and express the equation as one in x and v and hence find the general solution.

6.
$$\frac{dy}{dx} = -\frac{x^2+y^2}{2xy},\ y(1) = 1.$$

7.
$$(x^2+y^2)dx + (2xy + \cos y)dy = 0.$$

8. Solve
$$(y\cos x + 2xe^y) + (\sin x + x^2 e^y + 2)y' = 0$$
.

9.
$$2ydx + xdy = 0$$
.

10. By finding a suitable integrating factor, solve
$$(y^2 + xy)dx - x^2 dy = 0.$$

Problem Set 7 Linear and second order differential equations

Solve the following *first order linear differential equations.*

1.
$$x\frac{dy}{dx} + 3y = x^2,\ x > 0.$$

2.
$$\cosh x\frac{dy}{dx} + (\sinh x)y = e^{-x}.$$

3.
$$L\frac{di}{dt} + Ri = V.$$

4. Find the solution to the initial value problem

$$y' - 2xy = x,\ y(0) = 1.$$

5. Find the solution to the *second order differential equation*

$$y'' - 3y' - 4y = 2\sin x$$

with initial conditions, $y(0) = y'(0) = 1$.

6. Find a *particular solution* to

$$y'' - 3y' - 4y = 4x^2$$

.

7. Find a particular solution to

$$y'' - 3y' - 4y = e^{-x}$$

.

8. Solve using the substitution $p = \frac{dy}{dx}$ the equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2,$$

given the initial conditions that $y = 0$ and $y' = 1$ when $x = 1$.

9. By re-writing as a single second order differential equation, solve the *coupled system* of d.e.'s: $x_1' = x_1 + x_2$ and $x_2' = 4x_1 + x_2$.

10. Solve Question 9 by writing the system as a single matrix equation and diagonalizing the matrix.

Problem Set 8 Power series

1. Determine the centre and interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n}.$$

2. By integrating the geometric series for $\frac{1}{1+x^2}$, find the power series for $\arctan x$.
3. In a similar way, find the power series for

$$f(x) = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|.$$

The *Taylor series* for a function $f(x)$ around the point $x = x_0$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x-x_0)^n}{n!};$$

and the error in truncating the series after the term in $(x-x_0)^n$ is given by

$$E_n(x) = \frac{f^{(n+1)}(X)}{(n+1)!}(x-x_0)^{n+1}$$

for some X in the interval from x_0 to x.

4. Find the *McLaurin series* for $\sin x$ (Taylor series about 0) and hence find $\sin 3^\circ$ to an accuracy of 10^{-5}.

5. Find to three decimal places the value of

$$\int_0^1 e^{-t^2} dt$$

through the power series for $E(x) = \int_0^x e^{-t^2} dt$.
6. Using the error bound for the remainder, find how many terms of the series $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots$ are required to approximate e to an error less that 10^{-6}.

7. Find the *binomial series* for $f(x) = (1+x)^r$ ($r \in \mathbb{R}$) expanded about $x_0 = 0$ and show that it converges for $-1 < x < 1$.
8. Use Question 7 to find the Mclaurin series for $\frac{1}{\sqrt{1+x}}$.
9. Use Question 8 to find the power series expansion for $f(x) = \arcsin x$ for $-1 < x < 1$.
10. Find the McLaurin series for $\cos x$ by taking the real part of the series for e^{ix}.

Problem Set 9 Function of several variables

1. A point moves on the circle of the sphere $x^2+y^2+z^2=1$ where it meets the plane $x=\frac{2}{3}$ at what rate is z changing with y at the point $P(\frac{2}{3},\frac{1}{3},\frac{2}{3})$?

2. *Boyle's law* relate the Pressure, Tempertature and Volume of a gas by the equation

$$P=\frac{kT}{V}$$

where k is a constant. Show that if the temperature of the gas remains constant then

$$\frac{\partial P}{\partial V}+\frac{P}{V}=0.$$

3. Show equality of the mixed partial derivatives $f_{xy}=f_{yx}$ for the function $f(x,y)=e^{x-y}+\sin(xy)$.

4. Show that the function $f(x,y)=\ln(x^2+y^2)$ satisfies *Laplace's equation*

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0.$$

5. Show that the pair of functions $u(x,y)=\ln(x^2+y^2)$ and $v(x,y)=2\arctan\frac{y}{x}$ satisfy the *Cauchy-Riemann equations:*

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

6. Use the Chain rule to find $\frac{dw}{dt}$ given that $w=xy+z$, $x=\cos t$, $y=\sin t$, $z=t$ at $t=0$.

7. Given that $z=e^{xy}$, $x=2u+v$ and $y=\frac{u}{v}$ find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

8. Use the *total differential* (general form $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy+\cdots$) to estimate the change in percentage of the period of the simple pendulum

$$T=2\pi\sqrt{\frac{L}{g}},$$

if the length L of the pendulum increases by 2% and the acceleration due to gravity g decreases by $0\cdot 6\%$.

9. Find the *gradient vecto*r of the function $f(x,y)=\frac{x}{x^2+y^2}$ at the point $(2,3)$.

10. For the surface defined by the function of Question 9, find the *equation of the tangent plane* at the point $(2,3,\frac{2}{13})$.

Problem Set 10 Further partial differentiation and optimization

1. Find the *directional derivative* of $f(x, y) = 3x^2y$ at the point $(1, 2)$ in the direction of the vector $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$.

2. The temperature at position (x, y) in a region of the xy-plane is T° Celsius, where

$$T(x, y) = x^2e^{-y}.$$

In what direction at the point $(2, 1)$ does the temperature increase most rapidly?

3. Find the equation of the tangent plane and the equation of the normal line to the surface $z = x^2y$ at the point $(2, 1, 4)$.

4. What horizontal plane is tangent to the surface

$$z = x^2 - 4xy - 2y^2 + 12x - 12y - 1,$$

and what is its point of tangency?

5. Find all stationary points of

$$f(x, y) = x^3 + 3y^3 - \frac{1}{2}x^2 - 2x - 9y.$$

6. Classify each of the stationary points of Question 5 as a maximum, and minimum, or a saddle point of the function.

7. Find the closest point in the plane defined by the equation $z = 4 - x + 2y$ to the point $(-1, 3, 2)$ by treating the problem as one of optimization in two variables.

8. Use *Lagrange multipliers* to maximize x^3y^5 along the line $x + y = 8$.

9. Use Lagrange multipliers to find the dimensions of a rectangular open-topped box with volume 32m^3 with the least surface area.

10. Find the shortest distance from the origin to the curve $x^2y = 16$.

Hints for Problems

Problem Set 2

9. Express $F(m)$ in terms of $F(m-1)$.

Problem Set 3

1. Divide top and bottom by the highest power going and then move to the limit.
2. & 3. Use $\lim_{x\to 0} \frac{\sin x}{x} = 1$.
4. Multiply out the brackets.
5. Write as a quotient and then apply L'Hopital's rule.
6 & 7. Take logs and apply the result of Question 5.
8. $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.

Problem Set 4

2. Use the Chain rule.
3. Write down an equation for the separation, differentiate with respect to time and evaluate at the point using the information to hand.
5. Find the tangents of the tangents to the curve at the common point and then use the formula for $\tan(\theta_1 - \theta_2)$.
8. Put θ equal to the angle between the ground and the ladder and write the length of the ladder as a function of θ. Maximize this, noting that the maximum value only requires the values of $\cos\theta$ and $\sin\theta$ in order to be evaluated.
9. Write the ordinate of the position of the beam on the shore as a function of the angle of the ray. Apply the chain rule and evaluate appropriately.
10. Remember that $f^{-1}(x)'|_{f(x)} = \frac{1}{f(x)|_x}|$.

Problem Set 5

5. The arclength formula is $L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx$.

6. $\int_{t_0}^{t_1} \sqrt{(x'(t))^2 + (y'(t))^2 + (z(t))^2}\, dt$.

7. Evaluate the pair of integrals either side of the singularity, taking the limit in each case as the limit approaches the singularity.

9. Apply the result of Question 8 with $a = -1$.

10. Substitute $x = \frac{4}{t}$.

Problem Set 6

5. Use partial fractions.

6. Put $y = vx$.

7. An *exact* equation, meaning one of the form $M(x, y)dx + N(x, y)dy$ in which $\frac{\partial M}{\partial y} = \frac{\partial n}{\partial x}$ which has a solution $f(x, y) = C$ where $\frac{\partial f}{\partial x} = M(x, y)$ and $\frac{\partial f}{\partial y} = N(x, y)$.

9. This equation can be made exact by multiplying throughout by the *integrating factor* x.

10. $(xy^2)^{-1}$ is an integrating factor.

Problem Set 7

A *linear first order differential equation* is one that can be written in the form $y + p(x)y = q(x)$; multiplying by the integrating factor $\rho(x) = e^{\int p(x)dx}$ transforms the LHS into $(y\rho(x))'$.

7. If the RHS $p(x)$ is a solution to the corresponding homogeneous equation, then try $Axp(x)$ as the form of particular solution sought.

10. The solution of the system $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x} = C_1 e^{\lambda_1 t}\mathbf{e}_1 + C_2 e^{\lambda_2 t}\mathbf{e}_2$, where $\mathbf{e}_i$ is an eigenvector for the eigenvalue λ_i of A and the C_i are arbitrary constants.

Problem Set 8

4 & 5. Bound the error by using the fact that this is an alternating series.

10. Expand e^{ix} as an exponential series and use $\cos x =$Re(e^{ix}).

Problem Set 9

We use both types of notation for directional derivatives, $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$ etc. Notation extends to three and more variables in the obvious ways.

1. Write z as a function of x and y and find $\frac{\partial z}{\partial y}$.

6. $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial x}{\partial z}\frac{\partial z}{\partial t}$.

7. $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial y}$.

9. The definition of the gradient function is $\nabla f(x, y) = (f_x, f_y)$.

10. The tangent plane equation is $\nabla f\big|_{(x_0,y_0)} \bullet (\mathbf{x}-\mathbf{x_0}) = z - z_0$, where $\mathbf{x} = (x, y)$, $\mathbf{x_0} = (x_0, y_0)$.

Problem Set 10

1. The formula for the directional derivative in the direction of $\mathbf{u}$ is $D_{\mathbf{u}} f(x_0, y_0) = \nabla f\big|_{(x_0,y_0)} \bullet \hat{\mathbf{u}}$.

2. The function increases most rapidly in the direction of the gradient.

6. $D = D(x, y) = f_{xx}f_{yy} - (f_{xy})^2$; $D < 0$ indicates saddle point; $D > 0$ indicates minimum or maximum according as $f_{xx} > 0$ or $f_{xx} < 0$.

7 - 10. If $f(x, y, z)$ is the function to be optimized subject to $g(x, y, z) = 0$, solve $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ for unknown Lagrange multiplier λ: this condition ensures that the direction of maximum increase of $f(x, y, z)$ is normal to the surface $g(x, y, , z)$, which indicates an extreme point of $f(x, y, z)$ subject to the given constraint. This is also formulated via the Lagrangian $L(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z)$ where we then solve $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = 0$.

Answers to problems

Problem Set 1

1. $4(1-x)$. 2. $25 \cdot 13\text{cm}^3$. 3. $5 \cdot 1$. 4. $0 \cdot 693154 \cdots$ 5. 24. 6. $0 \cdot 0000326$ (3 sf). 7. $1 \cdot 00260$, $0 \cdot 00261$. 8. $1 \cdot 32471795724$ (10d.p.) 9. $0 \cdot 865474033102 \cdots$. 10. $0 \cdot 7391$ (4 d.p.)

Problem Set 2

1. $\frac{1}{2} \ln |\frac{1+x}{1-x}| + c$. 2. $\sinh^{-1} 2$. 3. 2π 4. $x = -1$, $x = 3, y = 2x$. 5. $\frac{3}{x-3} + \frac{2}{x+1}$. 6. $x^2 + 3 \ln |x-3| + 2 \ln |x+1| + c$. 7. 2. 8. $\ln 3$. 9. $m - 1$. 10. 8π .

Problem Set 3

1. $-\frac{1}{2}$. 2. 7. 3. 0. 4. 1. 5. 0. 6. 1 . 7. 1. 8. $\sqrt{e}$. 9. 0 . 10. $+\infty$.

Problem Set 4

1. $\frac{1}{\sqrt{x^2-1}}$. 2. $0 \cdot 256\pi$. 3. 50 mph. 4. 48 m^2/sec. 5. $45°$. 6. $-\frac{1}{2}$. 7. $-144\text{cm}/\text{sec}$. 8. $(1 + 2^{\frac{2}{3}})^{\frac{3}{2}}$. 9. $\frac{8\pi}{5}$ km/sec. 10. $\frac{1}{3}$.

Problem Set 5

1. $I_n = x^n e^x - nI_{n-1}$. 2. $e^x(x^3 - 3x^2 + 6x - 6) + c$. 3. $I = x(\sin^{-1} x)^n + n\sqrt{1-x^2}(\sin^{-1} x)^{n-1} - n(n-1) \int (\sin^{-1} x)^{n-2}\, dx$. 4. $x(\sin^{-1} x)^2 + 2\sqrt{1-x^2} - 2x$. 5. $\frac{\sinh^{-1}(2)+\sqrt{5}}{2}$. 6. $\frac{\pi}{2}\sqrt{a^2+b^2}$. 7. $3(2^{\frac{1}{3}})$. 9. 1. 10. $\frac{\pi\sqrt{3}\ln 2}{9}$.

Problem Set 6

1. $(\ln|t|+C)^{-1}$. 2. $\tan(e^x+C)$. 3. $y=\sqrt{2x+C}$. 4. $y=-\frac{1}{x+C}$. 5. $\frac{dx}{x}=\frac{1-v}{v^2-4}dv$, $|(y-2x)(y+2x)^3|=C$. 6. $x^3+3xy^2=4$. 7. $\frac{x^3}{3}+xy^2+\sin y=C$. 8. $y\sin x+x^2e^y+2y=C$. 9. $x^2y=C$. 10.$\ln|x|+\frac{x}{y}=c$; $x\neq 0, y\neq 0$.

Problem Set 7

1. $\frac{x^2}{5}+\frac{C}{x^3}$. 2. $-\frac{C-2e^{-x}}{e^x-e^{-x}}=\frac{Ce^x-2}{e^{2x}-1}$. 3. $\frac{V}{R}(1-e^{-\frac{Ri}{L}})$. 4. $\frac{3}{2}e^{x^2}-\frac{1}{2}$. 5. $\frac{36}{85}e^{4x}+\frac{2}{5}e^{-x}+\frac{3}{17}\cos x-\frac{5}{17}\sin x$. 6. $-x^2+\frac{3}{2}x-\frac{13}{8}$. 7. $-\frac{1}{5}xe^{-x}$. 8. $y(x)=\frac{x^3}{9}+\frac{2}{3}\ln|x|-\frac{1}{9}$. 9. & 10. $x_1(t)=Ae^{-t}+Be^{3t}$, $x_2(t)=-2Ae^{-t}+2Be^{3t}$.

Problem Set 8

1. $[-4,-1]$. 2. $=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots+\frac{(-1)^n x^{2n+1}}{2n+1}+\cdots$, $-1<x<1$. 3. $x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots$. 4. $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$; $0\cdot 05234$. 5. $0\cdot 747$. 6. $n=9$. 7. $1+\sum_{n=1}^{\infty}\frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}x^n, -1<x<1$. 8. $1+\sum_{n=1}^{\infty}\frac{(-1)^n 1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}x^n$; $-1<x\leq 1$. 9. ${}^{-1}x+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!(2n+1)}x^{2n+1}$. 10. $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$.

Problem Set 9

1. $-\frac{1}{2}$. 3. $f_{xy}=f_{yx}=-2e^{2x-y}+\cos xy-xy\sin xy$. 6. 2. 7. $\big(1+\frac{4u}{v}\big)e^{(2u+v)(u/v)}, -\frac{2u^2}{v^2}e^{(2u+v)(u/v)}$. 8. $1\cdot 6\%$. 9. $(\frac{5}{169},-\frac{8}{169})$. 10. $\frac{5}{169}(x-2)-\frac{8}{169}(y-3)=z-\frac{2}{9}$.

Problem Set 10

1. $\frac{48}{5}$. 2. $(2,-1)$. 3. $4x+4y-z=8; \frac{x-2}{4}=\frac{y-1}{4}=z-4$. $z=-31$. 5. 6. $\{(1,1)\text{ (minimum)}, (1,-1), (-\frac{2}{3},1), \text{saddle points} -\frac{2}{3}, -1\text{ maximum})\}$. 7. $(\frac{1}{2},0,\frac{7}{2})$. 8. $(3,5)$ the maximum. 9. $4\times 4\times 2$. 10. $2\sqrt{3}$.